Lecture Notes on Mathematical Modeling

Jerry Alan Veeh

August 8, 2003

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

§0. Preface

The objective of these notes is to provide the reader with some experience in mathematical modeling. Note the phrasing of the first sentence. Mathematical modeling is *not* a body of mathematical knowledge in the same way that Calculus or Differential Equations are, but rather a small collection of general principles which experience has proved to be helpful in the process of applying mathematical know-how to analyze problems that arise in various non-mathematical disciplines. Since the general principles are, at least retrospectively, self-evident, these notes will provide some instances in which these principles are applied to solve specific problems. The reader will then be asked to solve other, often unrelated, problems in order to gain experience in applying these principles. The exact mathematical methodology used in any particular instance is at the reader's discretion! Indeed, it is expected that the number of different solutions to some of the problems contained here will exceed the number of readers.

The structure of the remainder of these notes is as follows. Each section will contain an illustration of either a principle of mathematical modeling or a specific problem solving method. The illustration will usually be in the context of a specific example. There will be exercises within the section and it advisable for the reader to attempt to solve all of these. At the end of each section will be problems which are closely related to the ideas discussed in the section, but which may be more open ended. The reader should also attempt all of these. Solutions, often in the form of hints or useful facts, are provided for most of the problems. The laboratory sections consist of loosely worded problems similar to those that would be encountered in practice. These problems are the most time consuming, and the reader should solve these problems as time and interest permit. The use of computer software and calculators is often necessary to solve the problems given here.

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

§1. Modeling Using Physical Laws

Using known physical laws to construct mathematical models is one of the most widely used methods in the hard sciences and engineering. Some simple examples of this method are given here.

Example 1–1. A block of mass m is dropped from a height h above the ground. What is the motion of the block? The physics of motion is known to be governed by Newton's Law which states that the rate of change of momentum of an object at time t is equal to the net force on the object at time t. This physical law will play a central role in the solution of this problem.

The **first general principle** of mathematical modeling is to identify the known and unknown variables that are present in the problem. This is usually most easily accomplished by giving a full and carefully written verbal description of the problem that is to be solved. Attempting this verbal description will often demonstrate that the problem is not well understood. If the problem is being described by a person with minimal mathematical knowledge, it is only at this stage that the person will be able to contribute subject matter knowledge that may assist in model formulation and solution.

Example 1–2. In the previous example, the verbal description would be as follows. "Given the mass of a block and the height above the ground from which it is dropped, the velocity of the block and position of the block above the ground are to be found as functions of elapsed time since the block was dropped." With this verbal description in hand, the known variables are the mass m of the block and the height h from which it is dropped. The unknown variables are the velocity function $v(t)$ giving the downward velocity of the block at time t and the altitude function $A(t)$ giving the height of the block above the ground at time t. Having made these observations, re-examining the verbal description shows that $v(0) = 0$ is known too, and the convention has been introduced that time $t = 0$ corresponds to the time at which the block is dropped. Also $A(0) = h$.

Having identified the known and unknown variables in the problem at hand, the **second general principle** of mathematical modeling is to identify the relationships between the known and unknown variables in the model. In the present section, these relationships will arise from known physical laws. Generally, the relationships may arise from intuition based on knowledge of the subject matter area, empirical studies, or in other ways. At this stage, simplifying assumptions of various sorts are also often made. It is important that these assumptions be explicitly stated since the validity and accuracy of the model may depend heavily on the assumptions being satisfied.

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

Example 1–3. In the dropped block example the assumptions made are that air resistance is negligable, and that the gravitational force acting on the block is constant, independent of the height of the block above the ground. The mass of the block is also assumed to be constant. In this case, the net force acting on the block is the gravitational force mg where g is the gravitational constant ($g = 9.8$ meters per second per second). The momentum of the block at time t is $mv(t)$, by definition of momentum. Applying Newton's Law then gives $mv'(t) = mg$ as one equation relating the variables. The definition of velocity gives $A'(t) = -v(t)$ as another relation.

Exercise 1–1. Why does the minus sign enter into the expression for $A'(t)$?

At this point in the modeling exercise, the problem which was originally stated verbally has been given an entirely mathematical statement. Explicitly giving that mathematical statement and comparing it with the prior verbal statement provides a useful check on the progress of the modeling process at this point.

Example 1–4. The mathematical statement of the original problem is as follows. Given constants $m > 0$ and $h > 0$, and functions $v(t)$ and $A(t)$ satisfying $mv'(t) = mg$ with $v(0) = 0$ and $A'(t) = -v(t)$ with $A(0) = h$, find the functions $v(t)$ and $A(t)$ for $t \geq 0$.

With a purely mathematical statement of the problem in hand, any and all mathematical tools at the disposal of the problem solver can be used to find the solution to this mathematical problem.

Example 1–5. In the present case, the differential equations which make up the mathematical formulation of the problem can be easily solved to give $v(t) = gt$ and $A(t) = h - gt^2/2$.

Exercise 1–2. Verify that these are the correct solutions to the differential equations given earlier.

The solution of the mathematical problem is then interpreted in the context of the original problem. It is good practice to reiterate the assumptions that were made, and comment on how violations of the assumptions might affect the solution, if these effects are known. This is the **third general principal**: assess the effects of any assumptions that were made on the relationship between the variables in the model. This is an important but also very difficult step to carry out.

The verbal form of the solution should be understandable by anyone who could understand the verbal formulation of the problem that was obtained as the first step in the modeling process.

Example 1–6. At time $t > 0$, the downward velocity of the dropped block is $v(t) = gt$ and the height of the block above the ground is $h - gt^2/2$. These equations are valid for $0 \le t \le \sqrt{2h/g}$. The block hits the ground at time $t = \sqrt{2h/g}$ seconds. These results were obtained by neglecting the effect of air resistance, and assuming that the mass of the block is constant. The effect of violating either of these assumptions has not been assessed.

Exercise 1–3. How could you assess the impact of the assumptions on the solution?

The three principles of mathematical modeling illustrated here are

(1) Identify the known and unknown variables that are present in the problem.

(2) Identify the relationships between the known and unknown variables in the problem.

(3) Assess the effect of any assumptions made on the relationship between the variables in the problem.

Application of these principles is the core step in the modeling process, which may be summarized as follows.

(1) Give a full and carefully written verbal description of the problem that is to be solved.

(2) Identify the known and unknown variables present in the problem.

(3) Identify the relationships between the known and unknown variables in the problem.

(4) Explicitly state any assumptions that are made.

(5) Give a full and careful statement of the mathematical problem that is to be solved.

(6) Solve the mathematical problem.

(7) Give a verbal statement of the solution of the mathematical problem, including a discussion of the effect of any assumptions that were made.

Even in those cases in which the mathematical problem can not be solved, important information about the original problem can often be gained simply from the mathematical formulation of the problem.

Problems

Problem 1–1. If $f(x)$ is a function, state the definition of $f'(x)$.

Problem 1–2. State the Fundamental Theorem of Calculus.

Problem 1–3. A rectangular house is to be built with exterior walls that are 8 feet high. One wall of the house will face north. The total enclosed area of the house will be 1500 square feet. Annual heating costs for the house are determined as follows. Each square foot of exterior wall with a northern exposure adds \$4 to the annual heating cost; each square foot of exterior wall with an eastern or western exposure adds \$2 to the annual heating cost; each square foot of exterior wall with a southern exposure adds \$1 to the annual heating cost. Denote by L the length of each of the north and south facing walls, and by W the length of each of the east and west facing walls. Write an equation that expresses the total annual heating cost C in terms of L. For what value of L is the annual heating cost minimized, and what is the annual heating cost for this choice of L? The values given for the heating costs per square foot can only be known approximately. How does the solution depend on the given values for the heating costs?

Problem 1–4. Expand on the example of this section by considering the effects of air resistance. Do this by considering the resistive force to be proportional to $v(t)^\rho$ where $\rho > 0$ is a parameter to be determined by experiment.

Problem 1–5. A rubber ball rolls off the top of a table that is 1 meter high. With each bounce the ball loses 20% of its energy. Describe the motion of the ball until the time of the second bounce.

Problem 1–6. Suppose the block in the example is an asteroid that is within the earths gravitational field. Analyze the motion of the asteroid toward the earth. Neglect the effect of the sun, moon, and other planets. Hint: Here the gravitational force is given by Newton's Universal Law of Gravitation.

Problem 1–7. The frictional force between two contacting surfaces is often assumed to be proportional to the force pushing the two surfaces into contact. The constant of proportionality is called the **coefficient of friction** and is usually denoted by μ. Suppose a block of mass m made of one material is at rest on an incline plane made of another material. How is the coefficient of friction related to the angle of incline θ at which the block first starts to slide down the incline?

Problem 1–8. A pendulum is constructed by attaching a small ball of mass m at the end of a light inflexible rod of length l, which is then suspended from the ceiling at a frictionless pivot. The rod is intially displaced by an angle θ from the vertical, and then released. Describe the motion of the pendulum.

Problem 1–9. You sit in a swing suspended by ropes 2 meters long. Your initial angular displacement from vertical is $\pi/12$ radians and you give yourself an initial boost so that your angular velocity is 0.5 radians/sec. After the initial boost you make no additional effort to influence the swinging. What is the maximum angular displacement from the vertical? Neglect air resistance and friction. What happens if there is a frictional force proportional to the radial force along the ropes? What happens if there is an air resistance force that is proportional to velocity?

Solutions to Problems

Problem 1–1. $f'(x) = \lim_{y \to x} \dfrac{f(y) - f(x)}{y - x}$.

Problem 1–2. For any function $f(x)$ and any numbers $a < b$, $\int_a^b f'(x)\,dx = f(b) - f(a)$ and if $g(x)$ is any function for which $g'(x) = f'(x)$ for all x then $\int_a^b f'(x)\,dx = g(b) - g(a)$.

Problem 1–3. From the information given, $C = 8(4L + 2W + 2W + L) = 40L + 32W$. Since the area of the house is 1500 square feet, $LW = 1500$ so $W = 1500/L$. Hence $C = 40L + 32 \times 1500/L$. This formula is valid for $L > 0$. By making a reasonably careful graph of C versus L, the annual heating costs are minimized at about $L = 35$, which corresponds to an annual heating cost of about \$2900. One way to analyzed the effects of the heating costs on the solution would be to assume that the specified values are in the correct proportion, so that the true heating costs are $4p$, $2p$, and p respectively. The dependence of the solution on p could then be examined. There is always a trade-off between model accuracy and solvability.

Problem 1–4. The solution to the differential equation in this case exhibits quite different behavior. In particular, there is a finite terminal velocity, that is $\lim_{t\to\infty} v(t)$ exists as a finite positive number. What is the effect of ρ on the terminal velocity? What is the effect of the proportionality constant on the terminal velocity? What is the relationship between h and whether the terminal velocity is nearly attained?

Problem 1–5. There is a two dimensional equation of motion here. Symmetry governs the relationship between the trajectory of the ball immediately before and immediately after a bounce. The kinetic energy of the ball is $(1/2)mv^2$, where m is the mass and v the velocity of the ball.

Problem 1–6. Newton's Universal Law of Gravitation states that the graviational force between two bodies is proportional to the product of their masses and inversely proportional to the square of the distance between them. The proportionality constant is $G = 6.67 \times 10^{-11}$ cubic meters per second squared per kilogram. The mass of the earth is 5.98×10^{24} kilograms, and its diameter is 6.37×10^6 meters.

Problem 1–7. The force pushing the block and plane together is $mg \cos\theta$.

Problem 1–8. Under certain assumptions a simple differential equation for the angle $\theta(t)$ that the rod makes with the vertical can be derived. A phase diagram may prove useful in describing the motion.

Problem 1–9. Let m denote your mass. Since the displacement is small, the differential equation for the angular displacement from the vertical, $\theta(t)$, here is $2m\dfrac{d^2}{dt^2}\theta(t) = -mg \sin\theta(t)$. Using the approximation $\sin\theta(t) \approx \theta(t)$ under the assumption that $\theta(t)$ is small gives the solution as $\theta(t) = C_1 \cos(t\sqrt{g/2} + C_2)$

in phase amplitude form. The initial conditions give $\pi/12 = C_1 \cos C_2$ and $0.5 = -C_1\sqrt{g/2}\sin C_2$. Solving gives $C_1 = 0.345$ and $C_2 = -0.711$. The maximum angular displacement is therefore 0.345 radians or about 19.8 degrees. The radial force component is $mg\cos\theta(t)$, which contributes the frictional term.

Solutions to Exercises

Exercise 1–1. The coordinate system in which the analysis is taking place has the 'up' direction as positive. This should have been stated in the problem description.

Exercise 1–2. Simply plug in and check that these functions solve the differential equations.

Exercise 1–3. Qualitatively, the presence of air resistance would reduce the velocity of the block. If a specific model for the effect of air resistance was postulated, the effect could be studied by modifying the equation of motion. The effect of varying mass could also be studied by leaving a time dependent mass term in the equation of motion.

§2. The Method of Slicing

One method of obtaining relationships between variables is based on the idea of taking infinitesimal slices through time or space. The relationships that result by using this method are usually ordinary or partial differential equations. The method of slicing is an appropriate tool to use if the microscopic behavior of the phenomenon under consideration is simpler than its global behavior. Two examples are given here.

Example 2–1. How much work is done in pushing a block of mass m up a frictionless parabolic ramp? From elementary physics, the amount of work done in pushing a block a distance d in a straight line by applying a constant force F in the same direction as the motion is the product of the force and distance, Fd. In the present case, the path is not a straight line, so this physical law can not be used directly. The method of slicing applied here is based on the observation that over a short segment of the path, the path is very nearly straight and the force is very nearly constant. For concreteness, assume that the parabolic path is the section of the parabola $y = x^2$ for $0 \le x \le 10$. Denote by $W(x)$ the work done in sliding the block from the origin to the point (x, x^2) along the path. If z is slightly larger than x, $W(z) - W(x)$ is the work done in sliding the block along the nearly straight line path from (x, x^2) to (z, z^2). The distance between these two points is $\sqrt{(z-x)^2 + (z^2 - x^2)}$ by the distance formula. This short line segment makes an angle of $\arctan(2x)$ with the horizontal, so the gravitational force which must be overcome is $mg\sin(\arctan(2x)) = 2xmg/\sqrt{1+4x^2}$. Thus $W(z) - W(x) = \sqrt{(z-x)^2 + (z^2-x^2)}\,2xmg/\sqrt{1+4x^2}$, approximately, and this approximation becomes exact as z approaches x. Dividing both sides of this equation by $z - x$ and taking limits as $z \to x$ gives $W'(x) = 2xmg$. From this, the total amount of work done can be easily found.

Exercise 2–1. Complete the details of the preceding example.

Example 2–2. A perfectly elastic string is stretched along the x axis with its ends held fixed. At time $t = 0$ the string is plucked and then allowed to vibrate freely. What is the motion of the string? Denote by $H(x, t)$ the height of the string above (or below) the point x on the axis at time t. For fixed t, $H(x, t)$ gives the position of the entire string at that particular point in time; for fixed x, $H(x, t)$ describes the up and down motion of a particular point on the string through time. The method of slicing can be applied here in order to find a relationship that must be satisfied by the function $H(x, t)$. Suppose y is a little larger than x. Newton's Law applies to the short segment of string between the points $(x, H(x, t))$ and $(y, H(y, t))$. The mass of this segment of string is approximately $\rho\sqrt{(y-x)^2 + (H(y,t) - H(x,t))^2}$, where ρ is the constant density of the string. If the vibrations are of small magnitude, then this mass is approximately $\rho(y - x)$. The two forces acting on the string are the gravitational

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

force and the force arising from the tension in the string. To simplify the discussion, the gravitational force will be neglected. The left pointing tension force at x is a constant T times the unit vector in the direction of the force. Since the string is flexible, the direction of the tension force is parallel to the unit tangent vector to the string at x. This unit tangent vector is $-(1, H_1(x,t))/\sqrt{1+H_1(x,t)^2}$, where $H_1(x,t) = \frac{\partial}{\partial x}H(x,t)$. If the vibrations are of small magnitude, then $H_1(x,t)^2$ will be very small, and this tension force is approximately $-T(1, H_1(x,t))$. Similarly, the right pointing tension force at y is approximately $T(1, H_1(y,t))$. The sum of these two forces is the net force acting on the short segment of string. According to Newton's Law, this net force must be equal to the rate of change of the momentum of this short segment of string. Since the points of string move only in the vertical direction, applying Newton's Law to the vertical components of the force and momentum gives the equation $T(H_1(y,t) - H_1(x,t)) = \rho(y-x)\dfrac{\partial^2}{\partial t^2}H(x,t)$. Dividing both sides of this equation by $y-x$ and taking limits as $y \to x$ gives $T\dfrac{\partial^2}{\partial x^2}H(x,t) = \rho\dfrac{\partial^2}{\partial t^2}H(x,t)$. This gives the equation satisfied by $H(x,t)$.

Exercise 2–2. List all of the assumptions made in this derivation.

Problems

Problem 2–1. The slicing method can be used to obtain the usual formulas for volumes and surface areas of revolution. Here is a simple case. A candle holder was designed by revolving the curve $y = e^x$ about the x axis for $0 \le x \le 2$. Here x and y are measured in centimeters. This gives the basic form for the solid which makes up the holder. A cylindrical pocket of radius 1 centimeter and depth 1 centimeter is left in the small end (at $x = 0$) in order to hold the candle. The base of the holder is located at $x = 2$. What amount of material is required to make this (solid) candle holder? The visible surface of the holder will be plated with a layer of chromium which is 10^{-3} centimeters thick. Approximately what volume of chromium will be required to coat the visible surface of the candle holder?

Problem 2–2. A pail which is 30 cm high has a flat bottom and slanted sides. The radius of the top of the pail is 20 cm and the radius of the bottom of the pail is 15 cm. What is the volume of water that the pail can hold? Suppose that water flows into the pail at the constant rate of 1 liter per minute. Denote by $D(t)$ the depth of water in the pail at time t. Find a formula for $D(t)$ given that the pail is initially empty.

Problem 2–3. In the ramp problem, suppose that the coefficient of friction between the mass and the ramp is $\mu > 0$. Add such a frictional force and rework the analysis.

Problem 2–4. In the vibrating string problem, what relationship must be satisfied by $H(x, t)$ if the effect of gravity is taken into account?

Problem 2–5. It is a physical fact that the fluid pressure at a given depth in a liquid is the same in all directions. If the density δ of the fluid is same at all points in the fluid then the pressure at depth d is δd per unit area. The face of a dam is 300 meters wide and 100 meters tall. What is the total force exerted on the face of the dam by the water backed up behind it?

Problem 2–6. Suppose that the density of the fluid depends on the depth in the fluid. Rework the previous problem.

Problem 2–7. A dog and a rabbit are initially separated in a large open field. At time $t = 0$ the dog and rabbit see each other. The rabbit immediately begins running in a straight line with constant speed. The dog also begins running with constant speed. The dog runs in such a way that at any instant its direction of motion is directly toward the position of the rabbit at that instant. Describe the motion of the rabbit and the dog. Under what conditions does the dog catch the rabbit?

Solutions to Problems

Problem 2–1. Let $V(x)$ be the volume of material (before the pocket is formed) to form the holder from the origin to the point (x, e^x). Then if $j > x$, $V(j) = V(x) + \pi e^{2x}(j-x)$ so $V'(x) = \pi e^{2x}$. By the Fundamental Theorem, $V(2) - V(0) = \int_0^2 \pi e^{2x}\,dx = \pi(e^4 - 1)/2$. Now the pocket has volume π, so the total amount of material used is $\pi(e^4-3)/2 = 81.0501\text{cm}^3$. The surface area $S(x)$ of the holder up to x must satisfy the relation $S(y) = S(x) + 2\pi e^x\sqrt{(y-x)^2 + (e^y - e^x)^2}$ for $y > x$. Hence $S'(x) = 2\pi e^x\sqrt{1+e^{2x}}$. The volume is approximately $10^{-3}(S(2) - S(0)) = 10^{-3}\int_0^2 2\pi e^x\sqrt{1+e^{2x}}\,dx\,\text{cm}^3$.

Problem 2–2. Geometric considerations show that the radius of the pail h cm up from the bottom is $15 + h/6$. Thus if $V(h)$ is the volume of water in the pail when it is filled to a depth of h cm, and if $k > h$, then $V(k) = V(h) + \pi(15 + h/6)^2(k-h)$. Hence $V'(h) = \pi(15 + h/6)^2$. Using the Fundamental Theorem gives $V(30) - V(0) = \int_0^{30} V'(h)\,dh = \int_0^{30} \pi(15 + h/6)^2\,dh = \pi\int_{15}^{20} u^2 6\,du = 2\pi(20^3 - 15^3) = 9250\pi$ cubic centimeters. Suppose $t > s$. Then $D(t) = D(s) + \dfrac{1 \times (t-s)}{\pi(15 + D(s)/6)^2}$. Hence $D'(s) = \dfrac{1}{\pi(15 + D(s)/6)^2}$, or $(15 + D(s)/6)^2 \times D'(s) = 1/\pi$. Integrating both sides of this equation from 0 to t gives $(15 + D(t)/6)^3 - 15^3 = t/2\pi$, from which $D(t) = 6((t/2\pi + 15^3)^{1/3} - 15)$.

Problem 2–3. If the coefficient of friction is μ then the frictional component of the force along the short segment of the path is $\mu mg\cos(\arctan(2x))$. This force must also be overcome to move the block up the ramp, and thus adds to the total amount of work required.

Problem 2–4. The gravitational force of $\rho g(y-x)$ must be added at the point at which Newton's Law is applied.

Problem 2–5. Slice as a function of depth so that the pressure at that depth is constant. Pressure times area is force.

Problem 2–6. The pressure at depth d would then be $\int_0^d \delta(h)\,dh$, where $\delta(h)$ is the density at depth h.

Problem 2–7. Let $R(t)$ and $D(t)$ be the two dimensional vectors that represent the positon of the rabbit and dog at time t. From the information given $R(t) = R(0) + R'(0)t$ and $D'(t) = \| D'(0) \| (R(t) - D(t))/\| R(t) - D(t) \|$. This parametric description is not very useful. The path followed by the dog can also be described non-parametrically. Since the dogs speed is constant, there is a direct relationship between the arc length of the dogs path and the elapsed time. The information given reveals something about the slope of this path. Can you find this non-parametric description?

Solutions to Exercises

Exercise 2–1. From the Fundamental Theorem of Calculus, $W(x) - W(0) = \int_0^x W'(y)\,dy = \int_0^x 2ymg\,dy = mgx^2$. Since $W(0) = 0$, $W(x) = mgx^2$. Can you estimate the error in the approximate formula for $W(z) - W(x)$ and make the derivation rigorous?

§3. Laboratory 1

Write up full and complete solutions to the following problems. Keep the basic steps of mathematical modeling given earlier in mind. These steps are

(1) Give a full and carefully written verbal description of the problem that is to be solved.

(2) Identify the known and unknown variables present in the problem.

(3) Identify the relationships between the known and unknown variables in the problem.

(4) Explicitly state any assumptions that are made.

(5) Give a full and careful statement of the mathematical problem that is to be solved.

(6) Solve the mathematical problem.

(7) Give a verbal statement of the solution of the mathematical problem, including a discussion of the effect of any assumptions that were made.

Your writeup should take the form of a formal report.

1. Power is the rate at which work is done. How fast can a 200 horsepower engine move a 3500 pound car down the road? In the English system of measurement, power is measured in units of horsepower; in the metric system power is measure in watts. One watt is one newton meter per second (or one joule per second). Conversion factors: 1 horsepower is 746 watts; 1 kilogram is 2.2 pounds.

2. In the last part of Newton's Principia, he discusses the resistive force to motion through a rarefied medium. Newton's model is that the rarefied medium consists of small particles which are at rest. It is conceptually simpler to view the system as one in which the 'moving' body is at rest and the small particles are all traveling with the same velocity. When a particle strikes the body a fraction of the kinetic energy of the particle is transferred to the body. Compare the resistive force on a flat circular plate with that of a spherical ball of the same radius. How do these compare with the force on a paraboloid? What is the most aerodynamic shape?

3. The weight which can be supported by a cable is proportional to its cross sectional area. What is the most efficient design for a cable of fixed length that is to support a given mass suspended from the cable?

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

§4. Dimensional Analysis

Dimensional analysis is a method by which the general form of the relationship between variables can sometimes be found by using simple algebraic arguments.

Quantities that are measured typically have dimensions attached to them. A room has area A square meters; a rod is X inches long; a ball has mass b kilograms, etc. The same is true for the variables that arise in formulating a mathematical modeling problem. The objective is to use this dimensional information to get some idea of how the variables are related.

Example 4–1. Suppose two variables F and v appear in a modeling problem. Suppose F is a force and v a velocity. The relation $F = v$ is impossible on dimensional grounds. This is because force is measured in units of newtons (one newton is one kilogram meter per second per second), and velocity is measured in units of meters per second. Hence the left side of the equation involves units of mass which do not appear on the right side of the equation.

The example suggests that the dimensions of each variable appearing in the analysis should be expressed in terms of a standard system of dimensions in order to make such comparisons easier. Denote by M the dimension of mass; denote by L the dimension of distance; denote by T the dimension of time. The variable F of the example has dimensions MLT^{-2} while the variable v has dimensions LT^{-1}. So these two variables cannot possibly be equal. The general principle of dimensional analysis is that **two quantities can be equal if and only if the two quantities have the same dimensions.**

Example 4–2. Suppose that in addition to F and v a third variable q with dimensions MT^{-1} is also present. The equation $F = qv$ is then *possible*, based on dimensional arguments.

A simple observation in the last example is that the quantity $\dfrac{F}{qv}$ is **dimensionless**, that is, has no dimensions associated with it. The general principal implies that a dimensionless quantity must be a constant, since its value does not change when any of the units of measurement are changed. This suggests that identifying combinations of variables which are dimensionless could simplify the analysis of the problem. It is clear that the the only type of combination of the original variables that can produce a dimensionless variable is obtained by forming products of powers of the original variables.

Example 4–3. Suppose the problem is to find the air resistance force F on an object moving with velocity v. Intuition suggests that the cross sectional area A of the object in the direction of motion should also be an important variable. The

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

dimensions of F, v, and A are MLT^{-2}, LT^{-1}, and L^2 respectively. A product of powers of these variables of the form $F^a v^b A^c$ would have dimensions $M^a L^{a+b+2c} T^{-2a-b}$. This is dimensionless if and only if $a = 0$, $a + b + 2c = 0$, and $-2a - b = 0$. The only solution of these equations is $a = b = c = 0$, which produces no usable combination of the original variables. Since no combination of these variables is dimensionless, it is impossible for there to be any relationship involving only these variables which is independent of the system of measurement.

Exercise 4–1. Explain why there can not be a non-zero function H which does not depend on the system of measurement with $H(F, v, A) = 0$.

Example 4–4. The lack of a dimensionless combination of the variables F, v and A suggests that there is another variable which has been omitted from the problem formulation. One such possibility is the density δ of the air. It is easy to verify that $\dfrac{F}{\delta A v^2}$ is dimensionless, and hence a numerical constant, say k. This suggests the relationship $F = k\delta A v^2$ between the original variables.

Exercise 4–2. Set up equations as in the earlier example and verify that $\dfrac{F}{\delta A v^2}$ and its powers are the only dimensionless constants that are possible.

The linear equation method of identifying dimensionless quantities is completely general. Given any collection of variables a system of equations can be created which corresponds to the requirement that the product of powers of the variables is dimensionless. This system will have as many equations as there are fundamental dimensional units; the three fundamental dimensional units here are M, L, and T. The unknowns in this system will be the exponents of the variables; there will be as many unknowns in the system as there are variables. As in the example, this system of equations will always be a homogeneous system. The theory of linear systems states that a homogeneous system always has the solution in which the unknowns are all 0. The *number* of *linearly independent* solutions will be three less than the number of variables. Each set of linearly independent solutions will correspond to a choice of dimensionless constants. These observations constitute **Buckingham's Theorem**: The number of independent dimensionless constants which exist in a system of N variables and F fundamental dimensional units is $N - F$. Moreover, any relationship that exists between the variables in the problem and which does not depend on the system of units used must be expressible as a relationship between only these independent dimensionless constants.

Example 4–5. In the case of the simple frictionless pendulum, the period P should be related to the mass m, initial angle θ of displacement from the vertical, and length l of the pendulum. The gravitational constant g should also play a role. Since the angle θ is already dimensionless Buckingham's Theorem states that the

remaining 4 variables should combine into one new dimensionless variable. Simple computation shows that gP^2/l is dimensionless. Any relationship between these two dimensionless variables must be of the form $gP^2/l = f(\theta)$ for some numerical function f. Hence $P = h(\theta)\sqrt{l/g}$ for some numerical function h.

Exercise 4–3. What happened to the mass variable?

Problems

Problem 4–1. Show that if there is a relationship between energy E, mass m, and the speed of light c, then this relationship must be of the form $E = kmc^2$ for some dimensionless constant k.

Problem 4–2. The period P of pulsation of a pulsar is related to the star's radius r, its mass m, and the gravitational constant G found in Newton's Universal Law of Gravitation. Find the form of this relationship.

Problem 4–3. The hydrostatic pressure P of blood in the body is related to the density δ of blood, the height h of the heart above the lower body extremities, and the gravitational constant g. What is the form of this relationship? How does the hydrostatic blood pressure change as a person goes from a standing to a laying position?

Problem 4–4. In the pendulum model suppose the initial angular velocity ω is not zero. What is the form of the expression for the period of oscillation?

Problem 4–5. If a drag force F is incorporated into the pendulum model, how is the period determined? The drag force F may incorporate frictional as well as air resistance forces. What experiments could be done to investigate the validity of this model?

Problem 4–6. A raindrop falls from a motionless cloud. The terminal velocity v of the drop depends on its size, density, the gravitational constant, and the viscosity of air. Assume the drop is spherical with radius r, the density is ρ, and the viscosity of air is μ (which has units $ML^{-1}T^{-1}$). How is the terminal velocity related to the other quantities?

Problem 4–7. A crater is to be made using explosives. The hemispherical volume V of the crater is believed to depend on the depth h at which the explosive charge is located, the radius r of the explosive charge, the mass m of the explosive charge, the energy E of the explosive charge, the gravitational constant g, and the mass density ρ of the soil. How does the volume depend on these parameters?

Problem 4–8. One common rule for roasting a turkey is to set the oven at 400° and allow 20 minutes of cooking time per pound. Is this a good rule?

Solutions to Problems

Problem 4–1. The dimensions of energy are ML^2T^{-2}.

Problem 4–2. The dimensions of G are $M^{-1}L^3T^{-2}$.

Problem 4–3. Pressure is force per unit area.

Problem 4–4. The units of ω is T^{-1}.

Problem 4–5. There are now 3 dimensionless quantities.

Problem 4–6. There is one dimensionless quantity.

Problem 4–7. There are 6 quantities and 3 fundamental units, hence 3 dimensionless quantities.

Problem 4–8. What quantities should determine the cooking time? How many dimensionless quatities are there, and what are they?

Solutions to Exercises

Exercise 4–1. Since no combination of F, v, and A is dimensionless, suppose the measurement unit for force is changed arbitratily while the measurement units for velocity and area remain the same. The value of F then changes arbitrarily while the values of v and A remain fixed. Hence the function H does not depend on the value of F. Similarly, H doesn't depend on the value of v or A either. So H must be the zero function.

Exercise 4–2. The dimensions of $F^a \delta^b A^c v^d$ are $M^{a+b} L^{a-3b+2c+d} T^{-2a-d}$, and so this expression is dimensionless if and only if $a + b = 0$, $a - 3b + 2c + d = 0$, and $-2a - d = 0$. From here it follows that $b = -a$, $c = -a$, and $d = -2a$ and the dimensionless combination is $(F/\delta A v^2)^a$.

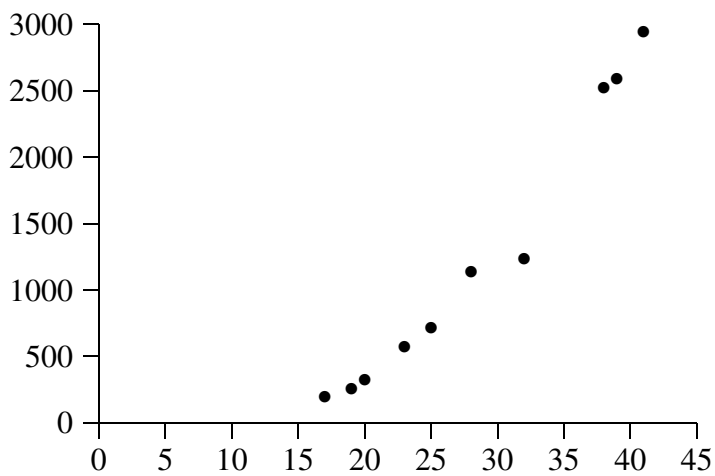
Exercise 4–3. Mass is simply not part of any dimensionless variable.

§5. Modeling Using Empirical Data

In many instances no physical laws can be readily identified in the context of the problem being studied. In such cases the general form of a *potential* relationship between the variables is all that can be obtained. The form of this relationship is often validated using data obtained from an experiment.

Example 5–1. Newton's Universal Law of Gravitation was formulated in order to summarize Kepler's Laws of planetary motion. These laws, in turn, were formulated to summarize extensive observations of planetary motion made by Tycho Brahe.

Example 5–2. Foresters would like to have a simple rule relating a measurement of a tree which can be easily made in the field to the number of board feet of lumber the tree will provide after harvesting. An attempt is made to obtain such a relationship using tree diameter as a predictor of finished lumber yield. One way of obtaining a possible model for experimentally obtained data is by making a plot of the data. The shape of the curve seen in the graph may suggest the form of the relationship between the variables. For the ponderosa pine data, the graph of the relationship between the number B of board feet of lumber in a tree and the tree diameter D in inches is as follows.



This plot suggests a power law relationship of the form $B = cD^p$ where c and p are unknown constants.

Exercise 5–1. Why would a straight line not provide a reasonable model?

Another way of obtaining a possible model is to use dimensional analysis.

Example 5–3. The yield B in board feet is a measure of the volume of usable material from a harvested tree. Thus B has units L^3. The diameter D has units of L. The height H of the tree would also appear to be an important variable.

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

The dimensions of H are also L. Dimensional analysis shows that B/HD^2 is the dimensionless combination of these variables. Hence $B = kHD^2$ is the form of a possible model here. If it is reasonable to assume that all trees to be harvested have about the same height, then the model becomes $B = cD^2$ for some (dimensional) constant c. The objective now is to assess the reasonableness of this model in view of the data.

Once a potential model is obtained, by dimensional analysis or other means, it should be tested against the experimental data for reasonableness. Usually the model will involve unknown parameters, such as the constant c in the example above. The experimental data is then used to do two things. First to estimate the unknown parameters in the model, and second to test the model for fit. Both of these objectives are typically achieved by using the method of least squares.

The **least squares principle** is that the best model for a given set of data will be the one for which the sum of squares of the deviations of the observed values from those predicted by the model is a minimum. This principle can be used to determine estimates of unknown constants in a given model, and also to compare the effectiveness of two different models in describing the same data.

Example 5–4. In the previous example the model $B = cD^2$ was suggested. The least squares principle says that the estimate of the constant c should be the value of c which minimizes the sums of squares of deviations between the observed values of B and those predicted by the model. For ponderosa pine the following data were obtained.

Diameter (inches)	Board Feet
17	190
19	250
20	320
23	570
25	710
28	1130
32	1230
38	2520
39	2590
41	2940

The sum of squares of these deviations is

$$(190 - c17^2)^2 + (250 - c19^2)^2 + \cdots + (2940 - c41^2)^2$$

and this is minimized when $c = 1.579$. This is the least squares estimate of c and the best fitting model of this form is $B = 1.579D^2$.

Using the least squares principal allows the unknown parameters in a model of a certain form to be estimated. A second question is whether even the best fitting model of the given form is an adequate model for the data. To answer this question the differences between the observed values and those predicted using the best fitting model are examined. These differences are called the **residuals**. If the best fitting model is appropriate the residuals should show no discernable pattern. Also the **error sum of squares** (or **residual sum of squares**), which is obtained by adding the squares of the residuals, should be small.

Exercise 5–2. Make a plot of the residuals for the ponderosa pine data using the best fitting model $B = 1.579D^2$. What is the error sum of squares?

The error sum of squares still seems rather large. One way of measuring the success of a proposed model in explaining the data is to compare the error sum of squares after fitting the model to the **total sum of squares** which is simply the sum of the squares of the observed values of the dependent variable. The ratio of the error sum of squares to the total sum of squares measures the fraction of the variation of the observations which is *not* explained by the model.

Example 5–5. In the previous example the total sum of squares is 25,521,900. So the fraction of the variablility in the data explained by the model is $1 - 753,477/25521900 = 0.97$. The proposed model explains about 97% of the variability in the data.

Exercise 5–3. Verify the computations in the preceding example.

Example 5–6. An alternate model is that the height of the tree is proportional to the diameter of the tree. This leads to the model $B = kD^3$. Using the data, the least squares estimate of k is 0.04362. The error sum of squares is $96,081.60$. Since the error sum of squares is much smaller than that for the earlier model, this model is better.

Exercise 5–4. Verify the computations in the preceding example. What are the residuals in this case?

Problems

Problem 5–1. In the lumber problem, does the model $B = cD^{\rho}$ give better results? How is ρ selected?

Problem 5–2. The following data gives the body mass, in grams, and pulse rate, in beats per minute, for various species of birds. What relationship is there between mass and pulse rate?

Bird	Mass	Pulse Rate
Canary	20	1000
Pigeon	300	185
Crow	341	378
Buzzard	658	300
Duck	1100	190
Hen	2000	312
Goose	2300	240
Turkey	8750	193
Ostrich	71000	65

Data from A. J. Clark, "Comparative Physiology of the Heart," Macmillan New York 1927 page 99.

Problem 5–3. It is believed that energy consumption has been increasing exponentially throughtout the past century. Below is a table giving energy consumption in various years, normalized so that the value for 1900 is 1. Is exponential growth a more reasonable model than linear growth? What is the annual growth rate?

Year	Consumption
1900	1.00
1910	2.01
1920	4.06
1930	8.17
1940	16.44
1950	33.12
1960	66.69
1970	134.29
1980	270.43
1990	544.67
2000	1096.63

Problem 5–4. The least squares method can be given a geometric interpretation. Suppose the observed data points are $(x_1, y_1), \ldots, (x_n, y_n)$ and define the n dimensional vectors $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$ and $U = (1, \ldots, 1)$. The least squares

line of the form $y = mx + b$ is found by determining m and b so that $\| Y - mX - bU \|$ is as small as possible. Now as m and b vary, $mX + bU$ sweeps out the two dimensional plane in n dimensional space that is spanned by the vectors X and U. The geometric interpretation of minimizing $\| Y - mX - bU \|$ is finding the point in this plane that is closest to the point Y. Argue that this occurs when m and b are chosen so that $Y - mX - bU$ is perpendicular to both X and U, and use this condition to find m and b.

Problem 5–5. The data given below are monthly mean thickness in Dobson units, one milli-centimetre ozone at standard temperature and pressure, of the ozone column at Arosa, Switzerland. Is there a simple model that summarizes this data?

Table 12.1
Monthly Mean Thickness in Dobson Units
of the Ozone Column at Arosa, Switzerland

Year	Jan	Feb	March	April	May	June	July	Aug	Sept	Oct	Nov	Dec
1926	-	-	-	-	-	-	312	300	281	267	295	325
1927	360	365	372	375	382	350	336	314	305	280	285	316
1928	324	325	347	359	373	331	308	295	288	268	279	313
1929	349	401	355	418	368	345	-	-	-	-	-	-
1930	-	-	-	-	-	-	291	299	-	-	-	-
1931	-	-	-	-	-	-	-	311	335	283	286	301
1932	318	347	370	394	360	347	334	299	292	287	293	281
1933	357	364	399	382	390	374	335	319	309	312	311	337
1934	334	321	392	358	365	355	328	321	282	287	291	297
1935	332	390	367	383	375	319	331	311	288	275	299	313
1936	329	393	398	384	373	352	328	315	303	310	298	307
1937	347	352	395	382	365	349	324	323	301	283	280	355
1938	337	370	325	392	384	336	325	325	296	280	285	299
1939	320	341	385	347	382	339	331	313	286	304	284	309
1940	387	400	418	430	403	388	346	323	310	292	302	338
1941	362	395	417	409	417	361	348	336	306	299	309	309
1942	400	422	373	408	376	347	325	309	284	272	298	313
1943	338	341	385	363	348	352	336	303	291	292	303	321
1944	300	365	385	360	349	351	319	306	290	293	298	320
1945	377	359	360	373	376	351	329	327	297	288	295	313
1946	336	352	380	361	355	344	318	307	276	291	297	316
1947	383	397	393	369	361	347	334	324	307	296	278	312
1948	341	371	348	374	353	345	345	311	299	281	286	321
1949	332	365	378	357	371	354	335	321	284	272	296	292
1950	352	365	365	382	374	354	322	316	292	288	287	340
1951	338	402	417	397	383	364	332	321	297	298	278	311
1952	378	384	411	386	385	359	341	320	317	297	302	332
1953	335	375	373	383	382	359	326	317	293	280	266	-
1954	-	-	373	415	389	362	348	329	305	285	291	284
1955	315	375	399	374	361	351	339	332	300	292	278	317
1956	341	402	381	395	365	360	327	308	287	284	286	312
1957	340	342	353	375	380	349	330	321	305	279	294	322
1958	361	351	411	417	369	369	349	330	306	312	305	326
1959	369	367	364	390	389	373	342	330	313	294	297	328
1960	349	397	405	400	382	353	339	315	306	299	284	332
1961	368	333	338	365	379	349	343	322	296	288	301	304
1962	352	362	428	400	363	349	338	302	296	271	298	309
1963	371	408	377	381	378	352	327	310	282	273	278	293
1964	301	335	347	378	363	330	326	323	299	288	274	310
1965	332	390	383	385	387	345	337	318	305	275	278	305
1966	354	338	401	380	372	-	-	326	307	282	321	319
1967	361	362	354	374	349	358	325	317	296	273	267	304
1968	342	383	376	379	349	351	337	335	308	275	270	331
1969	327	419	361	393	351	364	333	333	294	281	307	345
1970	331	417	414	419	389	360	331	323	291	278	271	309
1971	344	349	411	364	358	358	336	309	317	278	292	306

Problem 5–6. Below is the rainfall data for Los Angeles California 1877–1982

obtained from *The Los Angeles Times*. Annual rainfall totals July 1 through June 30. Is there a pattern?

Year	Rainfall	Year	Rainfall
1877-78	21.26	1930-31	12.53
78	11.35	31	16.95
79	20.34	32	11.88
1880-81	13.13	33	14.55
81	10.40	34	21.66
82	12.11	35	12.07
83	38.18	36	22.41
84	9.21	37	23.43
85	22.31	38	13.07
86	14.05	39	19.21
87	13.87	1940-41	32.76
88	19.26	41	11.18
89	34.84	42	18.17
1890-91	13.36	43	19.22
91	11.85	44	11.59
92	26.28	45	11.65
93	6.73	46	12.66
94	16.11	47	7.22
95	8.51	48	7.99
96	16.86	49	10.60
97	7.06	1950-51	8.21
98	5.59	51	26.21
99	7.91	52	9.46
1900-01	16.29	53	11.99
01	10.60	54	11.94
02	19.32	55	16.00
03	8.72	56	9.54
04	19.52	57	21.13
05	18.65	58	5.58
06	19.30	59	8.18
07	11.72	1960-61	4.85
08	19.18	61	18.79
09	12.63	62	8.38
1910-11	16.18	63	7.93
11	11.60	64	12.68
12	13.42	65	20.44
13	23.65	66	22.00
14	17.05	67	16.58
15	19.92	68	27.47
16	15.26	69	7.77
17	13.86	1970-71	12.32
18	8.58	71	7.17
19	12.52	72	21.26
1920-21	13.65	73	14.92
21	19.66	74	14.35
22	9.59	75	7.22
23	6.67	76	12.31
24	7.94	77	33.44
25	17.56	78	19.67
26	17.76	79	26.98
27	9.77	1980-81	8.98
28	12.66	81	10.71
29	11.52		

Solutions to Problems

Problem 5–1. Using the least squares principal here leads to a non-linear minimization problem which must be solved numerically.

Problem 5–2. One line of reasoning is this. Blood is circulated to provide the energy required to maintain body temperature. Heat loss is proportional to surface area of the body; mass is proportional to volume of the body. This leads to a model connecting pulse rate and mass. What is the form of that model? Hint: Try dimensional analysis.

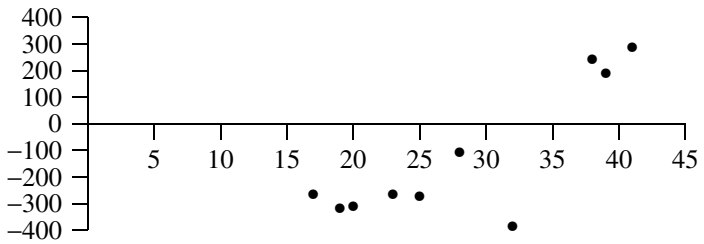
Problem 5–3. The form of an exponential growth model is ae^{bt}, and b is the rate of growth. A linear growth model is $a + bt$. For the exponential model the least squares principal leads to a non-linear minimization problem.

Problem 5–4. The vector $Y - mX - bU$ can be visualized as the line segment connecting the point $mX + bU$ in the plane to the point Y. This line segment is shortest when it is perpendicular to the plane. A vector is perpendicular to a plane exactly when it is perpendicular to the two generators of the plane. Now two vectors are perpendicular if and only if their dot product is zero. So $(Y - mX - bU)\bullet X = 0$ and $(Y - mX - bU)\bullet U = 0$, which gives two equations in the two unknowns m and b.

Solutions to Exercises

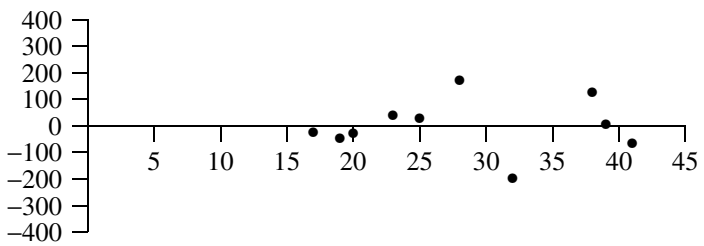
Exercise 5–1. Since trees of diameter 0 provide 0 wood, a straight line model would have to be a line through the origin. This clearly does not fit the data.

Exercise 5–2. The residual plot is as follows.



The model appears to overestimate yield systematically at small diameters and underestimate at large diameters. This suggests that the model could be improved. In this case the error sum of squares is 753, 477.27.

Exercise 5–4. The residual plot looks much more like a random pattern this time, as seen below.



§6. Difference Equations

Sometimes the independent variable can take on only discrete values. In such cases the use of differential equations is impossible. Difference equations are often well suited to such situations.

Example 6–1. The classical example in which a difference equation arises is the case of the Fibonacci sequence. Fibonacci postulated that each pair of rabbits that is at least two months old will produce another pair of rabbits each month. Denote by R_n the number of pairs of rabbits at the beginning of the nth month. If there is one pair of newborn rabbits at the beginning of the first month, $R_1 = 1$. By hypothesis, $R_2 = 1$ as well, since the rabbits must be two months old to reproduce. Then $R_3 = 2$, and generally, $R_n = R_{n-1} + R_{n-2}$, under the assumption that rabbits never die.

Exercise 6–1. Explain in detail why the equation $R_n = R_{n-1} + R_{n-2}$ holds under the assumption that rabbits never die.

An equation such as $R_n = R_{n-1} + R_{n-2}$ is called a **difference equation** or a **recursion** since the equation expresses the value of the sequence at n in terms of preceding values of the sequence. Given the starting values, the recursion can be used to compute as many terms of the sequence as desired.

Exercise 6–2. Given $R_1 = 1$ and $R_2 = 1$ use the recursion $R_n = R_{n-1} + R_{n-2}$ to compute R_n for $3 \le n \le 20$.

As in the case of differential equations, the existence of equilibrium solutions is usually the first item of interest in the study of difference equations. In the differential equations context, an equilibrium solution is a solution which has zero derivative, that is, is a constant; here an equilibrium solution is a solution whose difference in values over each single time epoch is zero, and hence is a constant.

Example 6–2. In the rabbit example, the only equilibrium solution is the zero solution, $R_n = 0$ for all n. It is also clear from the equation $R_n - R_{n-1} = R_{n-2}$ that zero is an unstable equilibrium: once the value of R differs from 0, the subsequent values of R will diverge to infinity.

Exercise 6–3. Explain why the subsequent values of R will diverge.

If a stable equilibrium exists, then the long term behavior of the sequence will most likely be known. This is often the important part of the analysis.

Example 6–3. A drug is administered to a patient every 4 hours. The patients metabolism causes the concentration of the drug in the patients blood to decrease by 75% between doses. The dose causes an instantaneous rise in blood concentration of

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

0.2 units. If C_n denotes the concentration of drug in the patients blood immediately after the nth dose, then $C_n = 0.25C_{n-1} + 0.2$. The equilibrium solution is $C_n = 0.2/0.75$ for all n, and this is a stable equilibrium. After many hours of medication, the concentration immediately after dosing will be about $0.2/0.75$.

In other cases, an explicit solution of the difference equation is important. By analogy with the differential equations case, for a homogeneous difference equation an exponential solution is attempted.

Example 6–4. For the Fibonacci sequence $R_n = R_{n-1} + R_{n-2}$, try a solution of the form $R_n = r^n$ for some number r which is to be determined. Then by substitution, r must satisfy $r^2 = r + 1$ in order for this to be a solution. Hence $r = \frac{1\pm\sqrt{5}}{2}$ are the two possible values of r. The general solution of the difference equation is $R_n = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n$, and the constants C_1 and C_2 are determined by the initial conditions $R_1 = 1 = R_2$.

Exercise 6–4. Find the constants C_1 and C_2 in this case.

Exercise 6–5. In the drug example, find the general solution. The equation $C_n = 0.25C_{n-1} + 0.2$ is a non-homogeneous difference equation, and one solution of this equation is known.

Problems

Problem 6–1. Suppose that in the rabbit model all rabbits die at exact age 6 months and immediately after giving birth. What is the model? Find the number of pairs of rabbits for each of the first 10 months.

Problem 6–2. A chain letter arrives. In the letter you are asked to send \$5 to each of the 5 people listed in the letter. Then you are to modify the letter by removing the top persons name from the list and adding your own name at the bottom of the list. Then you should mail copies of the modified letter to 5 people that you know. If L_k is the number of letters which have been sent by the time of generation k, what is L_k? How much money would you receive if all the people involved followed the intructions?

Problem 6–3. The mass $M(t)$ of a person at time t is affected by the caloric intake of the person, as well as the rate at which calories are burned by the metabolism and exercise. The person in question consumes food which provides 1600 calories of energy per day. Basic metabolic function uses 900 calories per day. Exercise consumes 20 calories per day per unit of body mass. Unused calories are stored by the body as fat; calories needed by the body in excess of those obtained through food are obtained from the fat store. Energy is stored in fat at the rate of 10,000 calories per kilogram of fat. Conversion of energy to and from fat is assumed to be perfectly efficient. According to this information, what is $M(t)$? What additional assumptions, if any, have you made? What is the equilibrium value of the mass for the person in question?

Problem 6–4. A loan in the amount of \$100,000 is to be repaid in 240 equal monthly installments. Assume the interest rate is 0.75% each month. What is the loan balance b_k immediately after the kth payment? What is the amount of each installment?

Problem 6–5. Simple games of chance can sometimes be at least partially analyzed using difference equations. In roulette, the wheel has numbers 1 through 36 together with 0 and 00, for a total of 38 numbers. Half of the 36 numbers are red and half are black, while 0 and 00 are green. A simple bet consists choosing one of the colors red or black and betting that the ball will land on a number of that color. If the ball comes to rest on the selected color, the payoff returns twice the original wager; otherwise the amount wagered is lost. If a person enters the casino with \$100 and bets \$1 on each spin of the wheel, what is the probability that the person will reach a total fortune of \$200 before losing everything? Let P_k denote the probability of reaching a total fortune of \$200 before losing everything if the person currently has k dollars. If the person currently has k, then after one play the person will either have $k+1$ or $k-1$. Hence $P_k = (18/38)P_{k+1} + (20/38)P_{k-1}$. Solve this equation for

P_k under the assumption that the person quits when her fortune reaches either 0 or \$200. What happens if the person bets \$5 each time?

Problem 6–6. Past experience indicates that 75% of the people who voted for the Republican presidential candidate in the last election will do so again in the present one; 25% will vote for the Democratic candidate. Among Democrats, 60% of those voting Democrat in the last election will do so again while 40% will vote Republican. What are the implications for the long run viability of the two party system?

Problem 6–7. An adequate supply of red blood cells is vital to survival. The spleen removes a fixed fraction of the supply of red blood cells each day. The bone marrow creates new red blood cells at a rate which is proportional to number removed by the spleen on the previous day. Does this mechanism lead to a steady supply of red blood cells in the body? Under what conditions will the system malfunction?

Solutions to Problems

Problem 6–1. The underlying principle is that the present number must equal the number at the beginning of the preceding month, plus the births, minus the deaths.

Problem 6–2. $L_{k+1} = 5L_k$ and presumably $L_0 = 1$.

Problem 6–3. Either a discrete or a continuous time model could be used here. In continuous time the equation is $10000 \times M'(t) = 1600 - 900 - 20M(t)$. In discrete time the equation is $10000M(t+1) = 10000M(t) + 1600 - 900 - 20M(t)$. It was assumed that the body always has sufficient fat store to provide for any energy deficiency. Once body fat is depleted, needed energy must in reality be provided from another source, which probably stores and releases energy at a different rate than fat. Setting $M'(t) = 0$ gives the equilibrium mass as 35 kg in the continuous model. Setting $M(t)$ equal to a constant gives the same equilibrium mass in the discrete time model. Are these equilibria stable?

Problem 6–4. If the amount of each installment is p then $b_{k+1} = 0.75b_k - p$. Notice that $b_0 = 100,000$ and $b_{240} = 0$.

Problem 6–5. The boundary conditions here are that $P_0 = 0$ and $P_{200} = 1$. Use these conditions to find the two arbitrary constants in the general solution of the difference equation.

Problem 6–6. The information gives a system of difference equations for R_n and D_n, the number voting for the Republican and Democratic candidate in the nth election.

Problem 6–7. If R_k is the red blood cell count on day k, S_k is the number of red blood cells removed by the spleen on day k, and M_k is the number of red blood cells created by the bone marrow on day k then $S_k = fR_k$ and $M_k = gfR_{k-1}$ where $0 < f < 1$ and $0 < g < 1$ are fixed numbers.

Solutions to Exercises

Exercise 6–1. The number of rabbits at time n is equal to the number alive at time $n-1$ plus the number of newborn rabbits at time n. The number of newborns is equal to the number of rabbits at time n that are at least 2 months old; such rabbits had to be alive at time $n-2$.

Exercise 6–2. The first few are 2, 3, 5, 8, 13.

Exercise 6–3. If $R_t > 0$ and $R_{t-1} = 0$ then from the equation $R_{t+1} = R_t > 0$. It follows that $R_{t+k} \geq kR_t$, so R diverges.

Exercise 6–4. $C_1 = 1/\sqrt{5}$ and $C_2 = -1/\sqrt{5}$.

Exercise 6–5. One solution is the equilibrium solution, so to find the general solution it is enough to find the general solution to the corresponding homogeneous equation $C_n = 0.25C_{n-1}$, which is $C_n = a(0.25)^n$. The general solution of the original equation is then $C_n = 0.2/0.75 + a(0.25)^n$.

§7. Laboratory 2

1. The demand for corn decreases as the current price for corn increases. The supply of corn in the current year depends primarily on the price of corn the farmers received during last years harvest. Corn is bought and sold at (approximately) the price at which supply and demand are in balance. What properties does the price of corn exhibit from year to year, if this model is correct?

2. A disease is spread by contact between a diseased person and a susceptible person. The disease confers immunity from reinfection by the disease. Immigrants who are all susceptible enter the population at a constant rate. How does the disease spread through the population?

3. National Income consists of consumer expenditures, private investment, and government expenditures. Consumer expenditures in the current quarter are a percentage of the national income in the previous quarter. Private investment is a percentage of the change in consumer expenditures between the current and previous quarter. Government expenditures are constant. What is the long term behavior of national income? What happens if the government follows a policy of increasing expenditures?

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

§8. Simulation of Deterministic Models

Sometimes the constructed model can not be analyzed with conventional tools. In such cases simulation is often used. The primary disadvantage of simulation is that the impact of the choice of values of unknown parameters is difficult to assess. The advantage of simulation is that once a model for the underlying process is obtained, the simulation can often be easily implemented on a computer.

Example 8–1. The simplest model for population growth is that the growth rate of the population is constant. If $P(t)$ is the population size at time t, this model implies that $P'(t)/P(t) = k$ where k is constant. This model is the exponential growth model, which implies that the population size grows to infinity as time passes.

Example 8–2. Since unlimited growth is not observed in any real process, the model of the previous example needs modification. One choice of modification would be to assume that the growth rate is a decreasing linear function of the population size. This leads to the equation $P'(t)/P(t) = a - bP(t)$ for positive constants a and b. This model is called the **logistic model**. The stable equilibrium of this model is the population size a/b, and given any positive initial population size this equilibrium size is (nearly) reached as time passes.

Exercise 8–1. Verify that a/b is a stable equilibrium for the logistic model.

Exercise 8–2. Solve the logistic equation and express $P(t)$ in terms of the initial population size and the constants a and b.

Example 8–3. The analog of the logistic model in discrete time is $(P_{n+1} - P_n)/P_n = a - bP_n$, where a and b are positive constants. This can be rewritten as $P_{n+1} = (1+a)P_n - bP_n^2$. As in the continuous model, the equilibrium solutions are 0 and a/b and a/b is a stable equilibrium. The behavior of this discrete time model is much more difficult to understand. Part of the reason for this is that there is no closed form solution for P_n. Another reason for this difficulty is that the solution of the discrete time model may overshoot the stable equilibrium. This cannot happen in the continuous time model.

Since there is no closed form solution to the difference equation of the preceding example, the behavior of the solution over short time periods cannot be determined without simply computing values. This makes it difficult to see exactly how a and b affect the behavior of the solution over the short term. This defect can only be overcome by computing the values of the solution for various (a, b) pairs and attempting to understand the influence of these parameters by examining the results of these computations.

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

Problems

Problem 8–1. To examine the possibility of overshoot of the stable equilibrium in the discrete logistic model one could proceed as follows. Define $f(x) = (1+a)x - bx^2$. If $f(x) \leq a/b$ for $0 \leq x \leq a/b$, then there would be no possibility of the solution overshooting the stable equilibrium at a/b. Does f obey this inequality?

Problem 8–2. Graph the solution of the discrete logistic equation for $a = 1$, $a = 2$, and $a = 3$ with $b = 0.1$ and $P_0 = 5$. Describe what happens in these cases.

Problem 8–3. Labor and Management are in a dispute over wages. Management's initial wage offer was M_0 dollars per hour, while the Labor negotiators wanted L_0 dollars per hour. It is reasonable to suppose that each offer will be updated by increasing or decreasing the previous offer by a fraction of the current difference in positions. Investigate the behavior of such a model. Will there be a wage agreement? If so, what will the amount of the wages be?

Problem 8–4. Suppose that a fish population follows the logistic population model, and the maximum carrying capacity of the habitat is 50,000 fish. The fish will be harvested by removing a fixed number H of fish in each time period. How should H be selected in order to maintain a stable supply of fish for the indefinite future?

Problem 8–5. One of the main food sources for large whales is krill. In uncrowded conditions in the absence of predators the krill population will double in a year. The maximum krill population is 400 tons per surface acre of ocean. At this maximum level of krill population, a blue whale population of 2000 will increase by an additional 2% per year, but the krill population will decrease by 25% per year. What is the behavior of the blue whale and krill populations if other predators are neglected?

Solutions to Problems

Problem 8–1. Since $f'(x) = (1 + a) - 2bx$, the maximum value of f occurs at $x = (1 + a)/2b$ and that maximum value is $(1 + a)^2/4b$. This maximum value is never smaller than a/b. The location of the maximum is in the interval $[0, a/b]$ if and only if $a \geq 1$. For $a < 1$ the maximum value of f on the interval $[0, a/b]$ occurs at $x = a/b$ and the maximum value is $(1 + a)(a/b) - b(a/b)^2 = a/b$. So f does obey this inequality for $0 < a < 1$.

Problem 8–3. The model could be expressed as $M_{n+1} = M_n + \alpha(L_n - M_n)$ and $L_{n+1} = L_n - \beta(L_n - M_n)$. The behavior of the negotiations can then be easily simulated. An exact solution is possible by eliminating L from the equations and the solving the single second order equation for M.

Problem 8–4. The difference equation for the population size of fish is $P_{n+1} = (1 + a)P_n - bP_n^2 - H$, and the equilibria can be easily found. One fixed point is stable for small enough H and the other is unstable.

Problem 8–5. One possible model is a system of difference equations in which B_n is the number of blue whales at the end of year n and K_n is the krill population in tons per surface acre at the end of year n. Once such model is $K_{n+1} = aK_n - bK_n^2 - cK_nB_n$ and $B_{n+1} = dB_n - eB_n^2 + fK_nB_n$ where a, b, c, d, e, f are positive constants. The condition that if there are no whales the maximum krill population is 400 means that one of the equilibrium solutions of the first equation must be 400 when $B_n = 0$ for all n. Since the equilibrium solutions are 0 and $(a - 1)/b$, $(a - 1)/b = 400$ or $a = 1 + 400b$. Since the krill population will double in one year (initially) under these same conditions, $a = 2$ (approximately). Plugging back in gives $b = 1/400$ (approximately). If there are no krill, the whale population must be 0. This means that the 2 equilibria of the second equation should be 0 if $K_n = 0$ for all n. Hence $(d - 1)/e = 0$ and $d = 1$. The condition that if there are 2000 whales and 400 tons per acre of krill now, then in one year there will be 2040 whales and 300 tons per acre of krill gives $300 = 2 \times 400 - (1/400)400^2 - c(400)(2000)$ and $2040 = 2000 - e(2000)^2 + f(400)(2000)$. From this, c can be found, and a relationship expressing e in terms of f can be obtained. This leaves one free parameter (e or f) in the model. The equilibria of the system of equations can then be found in terms of this free parameter. The stability of these equilibria can be studied.

Solutions to Exercises

Exercise 8–1. Here $P'(t) < 0$ if $P(t) > a/b$ while $P'(t) > 0$ if $P(t) > a/b$. So a/b is a stable equilibrium.

Exercise 8–2. $P(t) = \dfrac{ae^{ta}}{(a - bP(0))/P(0) + be^{ta}}$.

§9. Laboratory 3

1. Docking a space shuttle to a space station is usually carried out by automated systems. A standby manual system is in place in case the automated system fails. In the manual system thrusters fire for an adjustable length of time in order to provide acceleration/deceleration of the spacecraft. After firing the thrusters, a new closing velocity reading is obtained from other instruments; obtaining this new reading requires a certain minimum amount of time. The new thruster setting is then determined and manually set, and the thrusters are then fired. This process is repeated until docking occurs. Determine the conditions under which this docking procedure will be successful.

2. Two straight sections of train track are to be joined together by an intermediate section of track. Assume first that the two straight sections are parallel to each other, but are separated by a given distance. How should these sections of track be joined so that there are no shock forces imparted to the track as the train moves from the straight section onto the connecting section? What is the shortest length possible for the connecting section?

3. A simple pendulum of mass m is suspended at the end of a lightweight rod of length l as part of the mechanism for a grandfather clock. The spring mechanism of the clock provides a periodic driving force F. The resistive force on the pendulum is due to friction at the pivot and the coefficient of friction is μ. What value should F have and what should its frequency be in order to maintain an amplitude of $\pm 15^\circ$ for the pendulum?

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

§10. Modeling Using Randomness

In some cases randomness is part of the model. Some simple cases are examined here and simulation is used to study these processes.

Example 10–1. In an earlier example, medication was administered to a patient every 4 hours. In the time between doses, a certain fraction of the drug was metabolized and removed from the patients blood system. In practice the fraction that is metabolized will depend on the level of physical activity of the patient. If the patient is ambulatory, the variability can be rather pronounced. One way of incoporating this variablility into the model would be to assume that the fraction of medication that is metabolized is random. If C_n is the concentration of medication in the patients blood immediately after the nth dose is administered, the model could be $C_n = R_{n-1}C_{n-1} + 0.2$ where R_{n-1} is a random number representing the metabolism rate between doses $n-1$ and n. A simple model for R_{n-1} might be that it is a real number chosen at random between 0 and 0.5. This would reflect the belief that on the average 75% of the drug is metabolized between doses.

In order to simulate the blood concentration of drug, a way of generating the random numbers R_n appearing in the model is required. All programming languages and most software packages have routines for generating pseudo-random numbers. A **pseudo-random number** is a number that is approximately selected at random from the interval of real numbers from 0 to 1.

Exercise 10–1. If U is a pseudo-random number, what is the probability that $0 < U < 1/4$? What is the probability that $U > 5/8$?

Example 10–2. If U is a pseudo-random number then $U/2$ is a number selected at random from the interval of real numbers from 0 to $1/2$. Hence the random number generator provides a way of generating the random quantities R_n in the preceding example.

Example 10–3. In order to study the long run behavior of the concentration, the value of C_{100} will be simulated several times, and the values will be averaged. This will produce an estimate of the average value of C_{100} in the given scheme. One run of the simulation is conducted as follows. $C_1 = 0.2$ is known. A random number R_1 which is selected at random from the interval from 0 to 0.5 is obtained using the computer's random number generator. Using this $C_2 = R_1C_1 + 0.2$ is computed. A random number R_2 which is selected at random from the interval 0 to 0.5 is obtained. From this $C_3 = R_2C_2 + 0.2$ is computed. Finally, C_{100} is obtained.

The results of simulations are often displayed in a **histogram**. To make a histogram, the results are inspected and the largest and smallest observed values

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

are determined. The interval from the smallest to largest observed value is then subdivided into a convenient number of subintervals and the number of simulated values falling into each subinterval is counted. The histogram consists of a graph of these counts against the midpoint of the corresponding subinterval. The subinterval count divided by the total number of simulations gives an estimate of the probability that the outcome given by the model will lie in the subinterval.

Exercise 10–2. In the drug concentration example, simulate 5,000 values of C_{100} and make a histogram of the results. Compute the average of the simulated values of C_{100}. How does this compare with the value of the equilibrium?

Another common type of random experiment is a **Bernoulli trial**, which is simply a random experiment with only two possible outcomes. These outcomes are given the generic names success and failure.

Example 10–4. A coin is tossed. The outcome is either heads or tails. The coin toss constitutes a Bernoulli trial.

Example 10–5. You leave home each weekday morning to attend classes. You will either arrive safely or you won't. Each trip is a Bernoulli trial.

Bernoulli trials form a building block for a somewhat more complicated experiment. In a **binomial experiment** a fixed number n of Bernoulli trials is performed. Each of these trials has the same success probability p, and the outcome of each trial does not affect the probability of success or failure on any other trial. The number of successes in this sequence of Bernoulli trials is counted and this total is the result ascribed to the binomial experiment. In this setting the binomial experiment has parameters n and p.

Example 10–6. A coin is tossed 10 times and the number of heads in the 10 tosses is counted. This is a binomial experiment with parameters 10 and $1/2$.

Problems

Problem 10–1. How can the computer's random number generator be used to simulate the toss of a coin? Simulate tossing a coin 10 times and count the number of heads obtained in the 10 tosses. Repeat this simulation 10,000 times and make a graph of the number of times in these 10,000 repetitions there were 0, 1, . . . , 10 heads observed in the 10 simulated tosses. What would be your estimate of the probability of obtaining exactly 3 heads in 10 tosses of a coin?

Problem 10–2. How can you simulate the result of a binomial experiment with parameters n and p $(0 < p < 1)$? Perform 10,000 simulations of a binomial experiment with parameters 20 and 1/3 and make a histogram of the results. What is your estimate of the probability of seeing 4 or fewer successes in a binomial experiment with parameters 20 and 1/3?

Problem 10–3. How can a computer's random number generator be used to simulate the roll of a die? Simulate rolling a pair of dice and counting the total number of spots up on the two die. Repeat this simulation 10,000 times and make a graph of the number of times in these 10,000 rolls that the total on the two dice was 2, . . . , 12. What would be your estimate of the probability of obtaining a 6 on a single roll of 2 dice?

Problem 10–4. In the earlier roulette problem, a more complicated bet is the single number bet. In the single number bet, one number from 1 through 36 (or 0 or 00) is selected. If the ball lands on the selected number, a total amount of 36 times the original wager is returned to the bettor; otherwise the amount of the wager is lost. What is the probability that a player who begins with \$100 and bets \$1 on each play on a single number bet reaches a fortune of at least \$200 before losing everything?

Problem 10–5. In an earlier problem it was postulated that 75% of the people who voted for the Republican candidate in the last presidential election will do so in the present one, while 60% of those who voted for the Democratic candidate in the last election will do so in the present one. Interpret these statements to mean that each Republican voter is a Bernoulli trial with 'success' (= vote Republican again) probability 0.75, while each Democrat voter is a Bernoulli trial with success probability 0.60. In a community with 1,000 voters, what is the percentage of voters voting Republican over time? What happens if there are 100 voters in the community? 1,000,000 voters?

Problem 10–6. A disease is spread by contact between a diseased person and a susceptible person. The disease confers immunity from reinfection by the disease. Suppose each contact between a diseased and a susceptible person has a probability p of transmitting the disease to the susceptible person. The community of 10,000

persons experiences no immigration or emmigration, and the disease begins with 1 infected. How does the disease spread?

Solutions to Problems

Problem 10–1. Let U be a random number generated by the computer. If $U < 1/2$ say 'heads'; otherwise say 'tails'. Your estimate should be about 0.12.

Problem 10–2. Each Bernoulli trial could be simulated by saying 'success' if $U < 1/3$ and 'failure' otherwise. Your estimate should be about 0.15.

Problem 10–3. Let U be a random number generated by the computer. If $U < 1/6$ say '1 spot up on die'; if $1/6 < U < 2/6$ say '2 spots up on die'; etc. Your estimate should be about $5/36$.

Problem 10–4. A difference equation can be set up here as before, but it is rather difficult to solve. If one run of the experiment consists of the player beginning with \$100 and playing until either his fortune is 0 or at least \$200, then it is easy to simulate a large number of runs of this experiment.

Problem 10–5. In each election there are two binomial experiments: one determines the number of Republicans that stay Republican and the other determines the number of Democrats that crossover. Plot the total number of Republican voters in a long sequence of elections. Is there an equilibrium?

Problem 10–6. Conduct a large number of simulations for each of several selected values of p.

Solutions to Exercises

Exercise 10–2. I used subintervals of length 0.01 from 0.2 to 0.4 in making my histogram. The counts with each subinterval were all about 350 for the subintervals from 0.2 to 0.31, and the average was 0.262.

§11. Laboratory 4

1. The water tank for a small town is cylindrical in shape and is 57 feet in diameter and 40 feet high. A pump is used to maintain the amount of water in the tank between certain levels. When the level in the tank drops to 27 feet, the pump begins operation, and the pump stops when the level reaches about 35.5 feet. The table gives the water level in the tank at various times throughout a day. What is the water consumption for that day? Estimate the flow rate out of the tank at all times throughout the day.

Time (seconds)	Level (feet)
0	31.75
3316	31.10
6635	30.54
10619	29.94
13937	29.47
17921	28.92
21240	28.50
25223	27.97
28543	27.52
32284	26.97
35932	pump on
39332	pump on
39435	35.50
43318	34.45
46636	33.50
49953	32.60
53936	31.67
57254	30.87
60574	30.12
64554	29.27
68535	28.42
71854	27.67
75021	26.97
75021	pump on
82649	pump on
85968	34.75
89953	33.97
93270	33.40

2. The operator of a gasoline station does not want to run out of gas for her customers. The storage cost for storing the gas at her station is s dollars per day.

Copyright © 2003 Jerry Alan Veeh. All rights reserved.

It costs d dollars each time the delivery truck makes a delivery, and the maximum amount a truck can deliver is 10,000 gallons. The total storage capacity available is 50,000 gallons. Based on experience, the daily demand, in gallons, at her station for the past 100 days is as follows.

Demand (gallons)	Number of Days
1000-1099	1
1100-1199	2
1200-1299	5
1300-1399	12
1400-1499	20
1500-1599	27
1600-1699	18
1700-1799	8
1800-1899	4
1900-2000	3

How often should deliveries be scheduled, and how much should be ordered in each delivery?

3. All public places such as restaurants have a posted maximum occupancy. The maximum occupancy is determined by the requirement that the building must be quickly evacuated in case of a fire or other emergency. What is the maximum capacity of a restaurant for which the dining area is 1200 square feet, there is a single exit door 3 feet wide, and the room must be evacuated in 2 minutes in the case of an emergency?

§12. References

David Acheson, "From Calculus to Chaos," QA 303 A325 1997. Oxford University Press, Oxford England, 1997

Frank R. Giordano, Maurice D. Weir and William P. Fox, "A First Course in Mathematical Modeling," Second Edition. Brooks/Cole, New York, 1997

Roger Knobel, "An Introduction to the Mathematical Theory of Waves," American Mathematical Society, Rhode Island, 2000

J. David Logan, "Applied Mathematics," John Wiley and Sons, New York, 1987

Mark M. Meerschaert, "Mathematical Modeling," QA 401 M48 1993. Academic Press, New York, 1993

Michael Mesterton–Gibbons, "A Concrete Approach to Mathematical Modelling," QA 401 M5138 1995. John Wiley, New York, 1995

Ronald E. Mickens, "Difference Equations: Theory and Applications," QA 431 M426 1990. Van Nostrand Reinhold, New York, 1990

Frederic Y. M. Wan, "Mathematical Models and their Analysis," Harper and Row, New York, 1989

Paul Wilmott, Sam Howison, and Jeff Dewynne, "The Mathematics of Financial Derivatives," HG 6024 A3 W554 1995. Cambridge University Press, Cambridge England, 1995

Copyright © 2003 Jerry Alan Veeh. All rights reserved.