# Lecture Notes
# on the
# Mathematics of Finance

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## §0. Introduction

The objective of these notes is to present the basic aspects of the mathematics of finance, concentrating on the part of this theory most closely related to financial problems connected with life insurance. An understanding of the basic principles underlying this part of the subject will form a solid foundation for further study of the theory in other settings.

The central theme of these notes is embodied in the question, "What is the value today of a sum of money which will be paid at a certain time in the future?" Since the value of a sum of money depends on the point in time at which the funds are available, a method of comparing the value of sums of money which become available at different points of time is needed. This methodology is provided by the theory of interest.

Throughout these notes are various exercises and problems. The reader should attempt to work all of these. Also included are sample questions of the type likely to appear on the examination over this material given by the Society of Acturaries. These sample questions should be attempted after the material in the other exercises and problems has been mastered.

A calculator, such as the one allowed on the Society of Actuaries examinations, will be useful in solving the problems here. Proficiency in the use of the calculator to solve these problems is also an essential part of the preparation for the Society of Actuaries examination.

## §1. Elements of the Theory of Interest

A typical part of most insurance contracts is that the insured pays the insurer a fixed premium on a periodic (usually annual or semi–annual) basis. Money has time value, that is, \$1 in hand today is more valuable than \$1 to be received one year hence. A careful analysis of insurance problems must take this effect into account. The purpose of this section is to examine the basic aspects of the theory of interest. A thorough understanding of the concepts discussed here is essential.

To begin, remember the way in which compound interest works. Suppose an amount $A$ is invested at interest rate $i$ per year and this interest is compounded annually. After 1 year, the amount in the account will be $A + iA = A(1 + i)$, and this total amount will earn interest the second year. Thus, after $n$ years the amount will be $A(1 + i)^n$. The factor $(1 + i)^n$ is sometimes called the **accumulation factor**. If interest is compounded daily after the same $n$ years the amount will be $A(1+\frac{i}{365})^{365n}$. In this last context the interest rate $i$ is called the **nominal annual** rate of interest. The **effective annual rate of interest** is the amount of money that one unit invested at the *beginning* of the year will earn during the year, when the amount earned is paid at the *end* of the year. In the daily compounding example the effective annual rate of interest is $(1+\frac{i}{365})^{365} - 1$. This is the rate of interest which compounded annually would provide the same return. When the time period is not specified, both nominal and effective interest rates are assumed to be annual rates. Also, the terminology 'convertible daily' is sometimes used instead of 'compounded daily.' This serves as a reminder that at the end of a conversion period (compounding period) the interest that has just been earned is treated as principal for the subsequent period.

**Exercise 1–1.** What is the effective rate of interest corresponding to an interest rate of 5% compounded quarterly?

Two different investment schemes with two different nominal annual rates of interest may in fact be **equivalent**, that is, may have equal dollar value at any fixed date in the future. This possibility is illustrated by means of an example.

**Example 1–1.** Suppose I have the opportunity to invest \$1 in Bank A which pays 5% interest compounded monthly. What interest rate does Bank B have to pay, compounded daily, to provide an equivalent investment? At any time $t$ in years the amount in the two banks is given by $\left(1+\frac{0.05}{12}\right)^{12t}$ and $\left(1+\frac{i}{365}\right)^{365t}$ respectively. It is now an easy exercise to find the nominal interest rate $i$ which makes these two functions equal.

**Exercise 1–2.** Find the interest rate $i$. What is the effective rate of interest?

Situations in which interest is compounded more often than annually will arise frequently. Some notation is needed to discuss these situations conveniently. Denote by $i^{(m)}$ the nominal annual interest rate compounded $m$ times per year which is equivalent to the interest rate $i$ compounded annually. This means that

$$\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i.$$

**Exercise 1–3.** Compute $0.05^{(12)}$.

An important abstraction of the idea of compound interest is the idea of continuous compounding. If interest is compounded $n$ times per year the amount after $t$ years is given by $\left(1+\frac{i}{n}\right)^{nt}$. Letting $n \to \infty$ in this expression produces $e^{it}$, and this corresponds to the notion of instantaneous compounding of interest. In this context denote by $\delta$ the rate of instantaneous compounding which is equivalent to interest rate $i$. Here $\delta$ is called the **force of interest**. The force of interest is extremely important from a theoretical standpoint and also provides some useful quick approximations.

**Exercise 1–4.** Show that $\delta = \log(1+i)$.

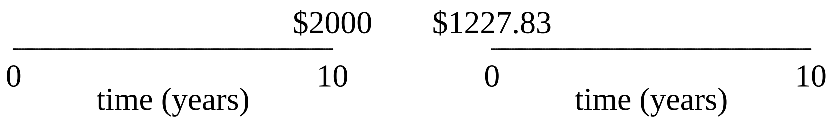
**Exercise 1–5.** Find the force of interest which is equivalent to 5% compounded daily.

The converse of the problem of finding the amount after $n$ years at compound interest is as follows. Suppose the objective is to have an amount $A$ $n$ years hence. If money can be invested at interest rate $i$, how much should be deposited today in order to achieve this objective? It is readily seen that the amount required is $A(1+i)^{-n}$. This quantity is called the **present value** of $A$. The factor $(1+i)^{-1}$ is often called the **discount factor** and is denoted by $v$. The notation $v_i$ is used if the value of $i$ needs to be specified.

**Example 1–2.** Suppose the annual interest rate is 5%. What is the present value of a payment of \$2000 payable 10 years from now? The present value is $\$2000(1+0.05)^{-10} = \$1227.83$.

The notion of present value is used to move payments of money through time in order to simplify the analysis of a complex sequence of payments. In the simple case of the last example the important idea is this. Suppose you were given the following choice. You may either receive \$1227.83 today or you may receive \$2000 10 years from now. If you can earn 5% on your money (compounded annually) you should be indifferent between these two choices. Under the assumption of an interest rate of 5%, the payment of \$2000 in 10 years can be replaced by a payment of \$1227.83 today. Thus the payment of \$2000 can be moved through time using the idea of present value. A visual aid that is often used is that of a time diagram which shows the time and amounts that are paid. Under the assumption of an interest rate of 5%, the following two diagrams are equivalent.

Two Equivalent Cash Flows




The advantage of moving amounts of money through time is that once all amounts are paid at the same point in time, the most favorable option is readily apparent.

**Exercise 1–6.** What happens in comparing these cash flows if the interest rate is 6% rather than 5%?

Notice too that a payment amount can be easily moved either forward or backward in time. A positive power of $v$ is used to move an amount backward in time; a negative power of $v$ is used to move an amount forward in time.

In an interest payment setting, the payment of interest of $i$ at the end of the period is equivalent to the payment of $d$ at the beginning of the period. Such a payment at the beginning of a period is called a **discount**. Formally, the **effective annual rate of discount** is the amount of discount paid at the *beginning* of a year when the amount invested at the *end* of the year is a unit amount. What relationship between $i$ and $d$ must hold for a discount payment to be equivalent to the interest payment? The time diagram is as follows.

Equivalence of Interest and Discount


The relationship is $d = iv$ follows by moving the interest payment back in time to the equivalent payment of $iv$ at time 0.

**Exercise 1–7.** Denote by $d^{(m)}$ the rate of discount payable $m$ times per year that is equivalent to a nominal annual rate of interest $i$. What is the relationship between $d^{(m)}$ and $i$? Between $d^{(m)}$ and $i^{(m)}$? Hint: Draw the time diagram illustrating the two payments made at time 0 and $1/m$.

**Exercise 1–8.** Treasury bills (United States debt obligations) pay discount rather than interest. At a recent sale the discount rate for a 3 month bill was 5%. What is the equivalent rate of interest?

The notation and the relationships thus far are summarized in the string of equalities

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(m)}}{m}\right)^{-m} = v^{-1} = e^{\delta}.$$

Another notion that is sometimes used is that of **simple interest**. If an amount $A$ is deposited at interest rate $i$ per period for $t$ time units and earns simple interst, the amount at the end of the period is $A(1 + it)$. Simple interest is often used over short time intervals, since the computations are easier than with compound interest.

The most important facts are these.

(1) Once an interest rate is specified, a dollar amount payable at one time can be exchanged for an equivalent dollar amount payable at another time by multiplying the original dollar amount by an appropriate power of $v$.

(2) The five sided equality above allows interest rates to be expressed relative to a convenient time scale for computation.

These two ideas will be used repeatedly in what follows.

## Problems

**Problem 1–1.** Show that if $i > 0$ then

$$d < d^{(2)} < d^{(3)} < \cdots < \delta < \cdots < i^{(3)} < i^{(2)} < i.$$

**Problem 1–2.** Show that $\lim_{m\to\infty} d^{(m)} = \lim_{m\to\infty} i^{(m)} = \delta$.

**Problem 1–3.** Calculate the nominal rate of interest convertible once every 4 years that is equivalent to a nominal rate of discount convertible quarterly.

**Problem 1–4.** Interest rates are not always the same throughout time. In theoretical studies such scenarios are usually modelled by allowing the force of interest to depend on time. Consider the situation in which \$1 is invested at time 0 in an account which pays interest at a constant force of interest $\delta$. What is the amount $A(t)$ in the account at time $t$? What is the relationship between $A'(t)$ and $A(t)$? More generally, suppose the force of interest at time $t$ is $\delta(t)$. Argue that $A'(t) = \delta(t)A(t)$, and solve this equation to find an explicit formula for $A(t)$ in terms of $\delta(t)$ alone.

**Problem 1–5.** Suppose that a fund initially containing \$1000 accumulates with a force of interest $\delta(t) = 1/(1+t)$, for $t > 0$. What is the value of the fund after 5 years?

**Problem 1–6.** Suppose a fund accumulates at an annual rate of simple interest of $i$. What force of interest $\delta(t)$ provides an equivalent return?

**Problem 1–7.** Show that $d = 1 - v$. Is there a similar equation involving $d^{(m)}$?

**Problem 1–8.** Show that $d = iv$. Is there a similar equation involving $d^{(m)}$ and $i^{(m)}$?

**Problem 1–9.** Show that if interest is paid at rate $i$, the amount at time $t$ under simple interest is more than the amount at time $t$ under compound interest provided $t < 1$. Show that the reverse inequality holds if $t > 1$.

**Problem 1–10.** Compute the derivatives $\frac{d}{di}d$ and $\frac{d}{dv}\delta$.

## Solutions to Problems

**Problem 1–1.** An analytic argument is possible directly from the formulas. For example, $(1 + i^{(m)}/m)^m = 1 + i = e^{\delta}$ so $i^{(m)} = m(e^{\delta/m} - 1)$. Consider $m$ as a continuous variable and show that the right hand side is a decreasing function of $m$ for fixed $i$. Can you give a purely verbal argument? Hint: How does an investment with nominal rate $i^{(2)}$ compounded annually compare with an investment at nominal rate $i^{(2)}$ compounded twice a year?

**Problem 1–2.** Since $i^{(m)} = m((1 + i)^{1/m} - 1)$ the limit can be evaluated directly using L'Hopitals rule, Maclaurin expansions, or the definition of derivative.

**Problem 1–3.** The relevant equation is $\left(1 + 4i^{(1/4)}\right)^{1/4} = \left(1 - d^{(4)}/4\right)^{-4}$.

**Problem 1–4.** In the constant force setting $A(t) = e^{\delta t}$ and $A'(t) = \delta A(t)$. The equation $A'(t) = \delta(t)A(t)$ can be solved by separation of variables to give $A(t) = A(0)e^{\int_0^t \delta(s)\,ds}$.

**Problem 1–5.** The amount in the fund after 5 years is $1000e^{\int_0^5 \delta(t)\,dt} = 1000e^{\ln(6)-\ln(1)} = 6000$.

**Problem 1–6.** The force of interest must satisfy $1 + it = e^{\int_0^t \delta(s)\,ds}$ for all $t > 0$. Thus $\displaystyle\int_0^t \delta(s)\,ds = \ln(1+it)$, and differentiation using the Fundamental Theorem of Calculus shows that this implies $\delta(t) = i/(1 + it)$, for $t > 0$.

**Problem 1–7.** $1 - d^{(m)}/m = v^{1/m}$.

**Problem 1–8.** $d^{(m)}/m = v^{1/m} i^{(m)}/m$.

**Problem 1–9.** The problem is to show that $1 + it > (1 + i)^t$ if $t < 1$, with the reverse inequality for $t > 1$. The function $1 + it$ is a linear function of $t$ taking the value 1 when $t = 0$ and the value $1 + i$ when $t = 1$. The function $(1 + i)^t$ is a convex function which takes the value 1 when $t = 0$ and $1 + i$ when $t = 1$.

**Problem 1–10.** $\dfrac{d}{di}d = \dfrac{d}{di}(1-1/(1+i)) = (1+i)^{-2}$, and $\dfrac{d}{dv}\delta = \dfrac{d}{dv}(-\ln(v)) = -v^{-1}$.

## Solutions to Exercises

**Exercise 1–1.** The equation to be solved is $(1 + 0.05/4)^4 = 1 + i$, where $i$ is the effective rate of interest.

**Exercise 1–2.** Taking $t^{th}$ roots of both sides of the equation shows that $t$ plays no role in determining $i$ and leads to the equation $i = 365((1+0.05/12)^{12/365}-1) = 0.04989$.

**Exercise 1–3.** $0.05^{(12)} = 12((1 + 0.05)^{1/12} - 1) = 0.04888$.

**Exercise 1–4.** The requirement for equivalence is that $e^{\delta} = 1 + i$.

**Exercise 1–5.** Here $e^{\delta} = (1 + 0.05/365)^{365}$, so that $\delta = 0.4999$. So as a rough approximation when compounding daily the force of interest is the same as the nominal interest rate.

**Exercise 1–6.** The present value in this case is $\$2000(1+0.06)^{-10} = \$1116.79$.

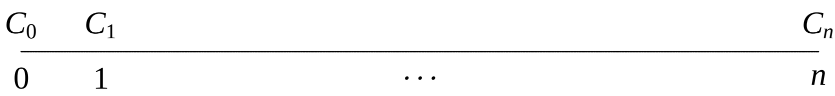
**Exercise 1–7.** A payment of $d^{(m)}/m$ made at time 0 is required to be equivalent to a payment of $i^{(m)}/m$ made at time $1/m$. Hence $d^{(m)}/m = v^{1/m}i^{(m)}/m$. Since $v^{-1/m} = (1+i)^{1/m} = 1+i^{(m)}/m$ this gives $d^{(m)}/m = 1-v^{1/m}$ or $1+i = (1-d^{(m)}/m)^{-m}$. Another relation is that $d^{(m)}/m - i^{(m)}/m = (d^{(m)}/m)(i^{(m)}/m)$.

**Exercise 1–8.** The given information is $d^{(4)} = 0.05$, from which $i$ can be obtained using the formula of the previous exercise as $i = (1 - 0.05/4)^{-4} - 1 = 0.0516$.

## §2. Cash Flow Valuation

Most of the remainder of these notes will consist of analyzing special cases of the following situation. Cash payments of amounts $C_0, C_1, \ldots, C_n$ are to be received at times $0, 1, \ldots, n$. A cash flow diagram is as follows.

A General Cash Flow




The payment amounts may be either postive or negative. A positive amount denotes a cash inflow; a negative amount denotes a cash outflow.

There are 3 types of questions about this general setting.

(1) If the cash amounts and interest rate are given, what is the value of the cash flow at a given time point?

(2) If the interest rate and all but one of the cash amounts are given, what should the remaining amount be in order to make the value of the cash flow equal to a given value?

(3) What interest rate makes the value of the cash flow equal to a given value?

The next several sections look at these questions in some typical settings. Here are a few simple examples.

**Example 2–1.** What is the value of this stream of payments at a given time $t$? The payment $C_j$ made at time $j$ is equivalent to a payment of $C_j v^{j-t}$ at time $t$. So the value of the cash flow stream at time $t$ is $\sum_{j=0}^{n} C_j v^{j-t}$.

**Example 2–2.** Instead of making payments of 300, 400, and 700 at the end of years 1, 2, and 3, the borrower prefers to make a single payment of 1400. At what time should this payment be made if the interest rate is 6% compounded annually? Computing all of the present values at time 0 shows that the required time $t$ satisfies the **equation of value** $300(1.06)^{-1} + 400(1.06)^{-2} + 700(1.06)^{-3} = 1400(1.06)^{-t}$, and the exact solution is $t = 2.267$. The **method of equated time** uses the approximation $v^a = e^{-\delta a} \approx 1 - \delta a$. In the present case this gives the approximate solution $(300(1) + 400(2) + 700(3))/1400 = 3200/1400 = 2.285$. The solution by the method of equated time is always larger than the exact solution. But as seen here, the solution by the method of equated time is much easier to compute, and is generally fairly accurate. This is especially interesting since the interest rate plays no role in the computations when the method of equated time is used.

**Example 2–3.** A borrower is repaying a loan by making payments of 1000 at the end of each of the next 3 years. The interest rate on the loan is 5% compounded annually. What payment could the borrower make at the end of the first year in order to extinguish the loan? If the unknown payment amount at the end of the year is $P$, the equation of value obtained by computing the present value of all payments at the end of this year is $P = 1000 + 1000v + 1000v^2$, where $v = 1/1.05$. Computation gives $P = 2859.41$ as the payment amount. Notice that the same solution is obtained using *any* time point for comparison. The choice of time point as the end of the first year was made to reduce the amount of computation.

## Problems

**Problem 2–1.** What rate of interest compounded quarterly is required for a deposit of 5000 today to accumulate to 10,000 after 10 years?

**Problem 2–2.** An investor purchases an investment which will pay 2000 at the end of one year and 5000 at the end of four years. The investor pays 1000 now and agrees to pay $X$ at the end of the third year. If the investor uses an interest rate of 7% compounded annually, what is $X$?

**Problem 2–3.** A loan requires the borrower to repay 1000 after 1 year, 2000 after 2 years, 3000 after 3 years, and 4000 after 4 years. At what time could the borrower make a single payment of 10000 according to the method of equated time? What is the exact time a payment of 10000 should be made if the interest rate is 4% effective?

**Problem 2–4.** A note that pays 10,000 3 months from now is purchased by an investor for 9500. What is the effective annual rate of interest earned by the investor?

**Problem 2–5.** A three year certificate of deposit carries an interest rate 7% compouned annually. The certificate has an early withdrawal penalty which, at the investor's discretion, is either a reduction in the interest rate to 5% or the loss of 3 months interest. Which option should the investor choose if the deposit is withdrawn after 9 months? After 27 months?

## Solutions to Problems

**Problem 2–1.** The equation of value is $5000(1+i/4)^{40} = 10000$, from which $i = 0.0699$.

**Problem 2–2.** The equation of value today is $2000v + 5000v^4 = 1000 + Xv^3$ where $v = 1/1.07$. Thus $X = 5773.16$.

**Problem 2–3.** The method of equated time gives the time of payment as $(1000 + 2\times 2000 + 3\times 3000 + 4\times 4000)/10000 = 3$ years, while the exact time $t$ is the solution of $1000v + 2000v^2 + 3000v^3 + 4000v^4 = 10000v^t$, where $v = 1/1.04$. Thus $t = 2.98$ years.

**Problem 2–4.** The equation involving the annual rate of interest $i$ is $9500(1+i)^{1/4} = 10000$, from which $i = 0.2277$.

**Problem 2–5.** The value of the note after 9 months at 5% interest is $(1.05)^{3/4} = 1.0372$, while the loss of 3 months interest would give the value of $(1.07)^{1/2} = 1.0344$, so the reduced interest rate is the preferred option. After 27 months the corresponding values are $(1.05)^{27/12} = 1.116$ and $(1.07)^{24/12} = 1.144$, so the forfeiture of 3 months interest is preferred.

## §3. Sample Question Set 1

**Solve the following 6 problems in no more than 30 minutes.**

**Question 3–1 .** Fund A accumulates at a force of interest $\frac{0.05}{1+0.05t}$ at time $t$ ($t \geq 0$). Fund B accumulates at a force of interest 0.05. You are given that the amount in Fund A at time zero is 1,000, the amount in Fund B at time zero is 500, and that the amount in Fund C at any time $t$ is equal to the sum of the amount in Fund A and Fund B. Fund C accumulates at force of interest $\delta_t$. Find $\delta_2$.

**A.** $\frac{31}{660}$

**B.** $\frac{21}{440}$

**C.** $\frac{1+e^{0.1}}{22+20e^{0.1}}$

**D.** $\frac{2+e^{0.1}}{44+20e^{0.1}}$

**E.** $\frac{2+e^{0.1}}{22+20e^{0.1}}$

**Question 3–2 .** Gertrude deposits 10,000 in a bank. During the first year the bank credits an annual effective rate of interest $i$. During the second year the bank credits an annual effective rate of interest ($i$−5%). At the end of two years she has 12,093.75 in the bank. What would Gertrude have in the bank at the end of three years if the annual effective rate of interest were ($i$ + 9%) for each of the three years?

**A.** 16,851

**B.** 17,196

**C.** 17,499

**D.** 17,936

**E.** 18,113

**Question 3–3 .** Fund $X$ starts with 1,000 and accumulates with a force of interest $\delta_t = \frac{1}{15-t}$ for $0 \leq t < 15$. Fund $Y$ starts with 1,000 and accumulates with an interest rate of 8% per annum compounded semi-annually for the first three years and an effective interest rate of $i$ per annum thereafter. Fund $X$ equals Fund $Y$ at the end of four years. Calculate $i$.

**A.** 0.0750

**B.** 0.0775

**C.** 0.0800

**D.** 0.0825

**E.** 0.0850

**Question 3–4**. Jeff puts 100 into a fund that pays an effective annual rate of discount of 20% for the first two years and a force of interest of rate $\delta_t = \frac{2t}{t^2+8}$, $2 \le t \le 4$, for the next two years. At the end of four years, the amount in Jeff's account is the same as what it would have been if he had put 100 into an account paying interest at the nominal rate of $i$ per annum compounded quarterly for four years. Calculate $i$.

**A.** 0.200

**B.** 0.219

**C.** 0.240

**D.** 0.285

**E.** 0.295

**Question 3–5** . On January 1, 1980, Jack deposited 1,000 into Bank X to earn interest at the rate of $j$ per annum compounded semi-annually. On January 1, 1985, he transferred his account to Bank Y to earn interest at the rate of $k$ per annum compounded quarterly. On January 1, 1988, the balance at Bank Y is 1,990.76. If Jack could have earned interest at the rate of $k$ per annum compounded quarterly from January 1, 1980 through January 1, 1988, his balance would have been 2,203.76. Calculate the ratio $k/j$.

**A.** 1.25

**B.** 1.30

**C.** 1.35

**D.** 1.40

**E.** 1.45

**Question 3–6** . You are given two loans, with each loan to be repaid by a single payment in the future. Each payment includes both principal and interest. The first loan is repaid by a 3,000 payment at the end of four years. The interest is accrued at 10% per annum compounded semi-annually. The second loan is repaid by a 4,000 payment at the end of five years. The interest is accrued at 8% per annum compounded semi-annually. These two loans are to be consolidated. The consolidated loan is to be repaid by two equal installments of $X$, with interest at 12% per annum compounded semi-annually. The first payment is due immediately and the second payment is due one year from now. Calculate $X$.

**A.** 2,459

**B.** 2,485

**C.** 2,504

**D.** 2,521

**E.** 2,537

## Answers to Sample Questions

**Question 3–1 .** The amount in fund A at time $t$ is $A(t) = 1000e^{\int_0^t \frac{0.05}{1+0.05s}\,ds} = 1000+50t$, the amount in fund B at time $t$ is $B(t) = 500e^{0.05t}$ and the amount in fund $C$ at time $t$ is $C(t) = A(t) + B(t)$. So $\delta_2 = C'(2)/C(2) = (A'(2) + B'(2))/(A(2) + B(2)) = (50 + 25e^{0.1})/(1100 + 500e^{0.1}) = (2 + e^{0.1})/(44 + 20e^{0.1})$. **D**.

**Question 3–2 .** From the information given, $10000(1 + i)(1 + i - 0.05) = 12093.75$, from which $i = 0.125$. Thus $10000(1 + i + 0.09)^3 = 17,936.13$. **D**.

**Question 3–3 .** After 4 years the amount in fund X is $15000/(15 - 4) = 15000/11$ and the amount in fund Y is $1000(1.04)^6(1+i)$. Equating these two gives $i = 0.0777$. **B**.

**Question 3–4 .** The amount Jeff actually has is $100(1 - 0.20)^{-2}e^{\int_2^4 \delta_t\,dt} = 312.50$, while what he would have under the other option is $100(1 + i/4)^{16}$. Equating and solving gives $i = 0.2952$.**E**.

**Question 3–5 .** The given information gives two equations. First, $1000(1+j/2)^{10}(1+k/4)^{12} = 1990.76$ and second $1000(1+k/4)^{32} = 2203.76$. The second gives $k = 0.10$, and using this in the first gives $j = 0.0799$. Thus $k/j = 1.25$. **A** .

**Question 3–6 .** From the information given, $3000(1.05)^{-8} + 4000(1.04)^{-10} = X + X(1.06)^{-2}$, from which $X = 2504.12$. **C** .

## §4. Annuities, Amortization, and Sinking Funds

Many different types of financial transactions involve the payment of a fixed amount of money at regularly spaced intervals of time for a predetermined period. Such a sequence of payments is called an **annuity certain** or, more simply, an **annuity**. A common example is loan payments. It is easy to use the idea of present value to evaluate the worth of such a cash stream at any point in time. Here is an example.

**Example 4–1.** Suppose you have the opportunity to buy an annuity, that is, for a certain amount $A > 0$ paid by you today you will receive monthly payments of \$400, say, for the next 20 years. How much is this annuity worth to you? Suppose that the payments are to begin one month from today. Such an annuity is called an **annuity immediate** (a truly unfortunate choice of terminology). The cash stream represented by the annuity can be visualized on a time diagram.

An Annuity Immediate

| | | | | | | |
|---|---|---|---|---|---|---|
| | \$400 | \$400 | ··· | ··· | \$400 | \$400 |
| 0 | 1 | 2 | | | 239 | 240 |

Clearly you would be willing to pay today no more than the present value of the total payments made by the annuity. Assume that you are able to earn 5% interest (nominal annual rate) compounded monthly. The present value of the payments is

$$\sum_{j=1}^{240}\left(1+\frac{.05}{12}\right)^{-j} 400.$$

This sum is simply the sum of the present value of each of the payments using the indicated interest rate. It is easy to find this sum since it involves a very simple geometric series.

**Exercise 4–1.** Evaluate the sum.

Since expressions of this sort occur rather often, actuaries have developed some special notation for this sum. Write $a_{\overline{n}|}$ for the present value of an annuity which pays \$1 at the *end* of each period for $n$ periods.

The Standard Annuity Immediate

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | | 1 | 1 | 0 |
| 0 | 1 | ··· | $n-1$ | $n$ | $n+1$ |

Then

$$a_{\overline{n}|} = \sum_{j=1}^{n} v^j = \frac{1-v^n}{i}$$

where the last equality follows from the summation formula for a geometric series. The interest rate per period is usually not included in this notation, but when such information is necessary the notation is $a_{\overline{n}|i}$. The present value of the annuity in the previous example may thus be expressed as $400a_{\overline{240}|.05/12}$.

A slightly different annuity is the **annuity due** which is an annuity in which the payments are made starting immediately. The notation $\ddot{a}_{\overline{n}|}$ denotes the present value of an annuity which pays \$1 at the *beginning* of each period for $n$ periods.

The Standard Annuity Due

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | | 1 | 0 | 0 |
| 0 | 1 | $\cdots$ | $n-1$ | $n$ | $n+1$ |

Clearly

$$\ddot{a}_{\overline{n}|} = \sum_{j=0}^{n-1} v^j = \frac{1-v^n}{d}$$

where again the last equality follows by summing the geometric series. Note that $n$ still refers to the number of payments. If the present time is denoted by time 0, then for an annuity immediate the last payment is made at time $n$, while for an annuity due the last payment is made at time $n-1$, that is, the beginning of the $n$th period. It is quite evident that $a_{\overline{n}|} = v\,\ddot{a}_{\overline{n}|}$, and there are many other similar relationships.

**Exercise 4–2.** Show that $a_{\overline{n}|} = v\,\ddot{a}_{\overline{n}|}$.

The connection between an annuity due and an annuity immediate can be viewed in the following way. In an annuity due the payment for the period is made at the beginning of the period, whereas for an annuity immediate the payment for the period is made at the end of the period. Clearly a payment of 1 at the end of the period is equivalent to the payment of $v = 1/(1+i)$ at the beginning of the period. This gives an intuitive description of the equality of the previous exercise.

Annuity payments need not all be equal. Here are a couple of important special modifications.

**Example 4–2.** An **increasing annuity immediate** with a term of $n$ periods pays 1 at the end of the first period, 2 at the end of the second period, 3 at the end of the third period, $\ldots$, $n$ at the end of the $n$th period. What is $(Ia)_{\overline{n}|}$, the present value of such an annuity? From the definition, $(Ia)_{\overline{n}|} = \sum_{j=1}^{n} jv^j$. Although this is not a geometric series, the same technique can be used. This procedure gives $(Ia)_{\overline{n}|} - v(Ia)_{\overline{n}|} = v + v^2 + \ldots + v^n - nv^{n+1}$ which gives $(Ia)_{\overline{n}|} = (a_{\overline{n}|} - nv^{n+1})/(1-v) = (\ddot{a}_{\overline{n}|} - nv^n)/i$.

**Exercise 4–3.** A **decreasing annuity immediate** with a term of $n$ periods pays $n$ at the end of the first period, $n-1$ at the end of the second period, $n-2$ at the end of the third period, ..., 1 at the end of the $n$th period. Find $(Da)_{\overline{n|}}$, the present value of such an annuity.

**Exercise 4–4.** An annuity immediate with $2n-1$ payments pays 1 at the end of the first period, 2 at the end of the second, ..., $n$ at the end of the $n$th period, $n-1$ at the end of the $n+1$st period, ..., 1 at the end of the $2n-1$st period. What is the present value of this annuity?

**Example 4–3.** A **deferred annuity** is an annuity in which the payments start at some future time. A standard deferred annuity immediate in which payments are deferred for $k$ periods, has the first payment of 1 made at time $k+1$, that is, at the end of year $k+1$. Notice that from the perspective of a person standing at time $k$, this deferred annuity immediate looks like a standard $n$ period annuity immediate. The present value of a $k$ year deferred, $n$ year annuity immediate is denoted ${}_{k|}a_{\overline{n|}}$. The present value of the deferred annuity *at time k* is $a_{\overline{n|}}$. Bringing this to an equivalent value at time 0 gives ${}_{k|}a_{\overline{n|}} = v^k a_{\overline{n|}}$. A time diagram shows that the deferred payments can be obtained by paying back payments that are received during the first $k$ periods. Thus ${}_{k|}a_{\overline{n|}} = a_{\overline{n+k|}} - a_{\overline{k|}}$.

**Exercise 4–5.** What is ${}_{k|}\ddot{a}_{\overline{n|}}$?

Theoretically, an annuity could be paid continuously, that is, the annuitant receives money at a constant rate of 1 dollar per unit time. The present value of such an annuity that pays 1 per unit time for $n$ time periods is denoted by $\bar{a}_{\overline{n|}}$. Now the value at time 0 of such a continuously paid annuity can be computed as follows. The value of the $dt$ dollars that arrive in the time interval from $t$ to $t+dt$ is $v^t\,dt = e^{-\delta t}\,dt$. Hence $\bar{a}_{\overline{n|}} = \int_0^n e^{-\delta t}\,dt = \dfrac{1-v^n}{\delta}$.

Annuity payments can be made either more or less often than interest is compounded. In such cases, the equivalent rate of interest can be used to most easily compute the value of the annuity.

**Example 4–4.** The symbol $a^{(m)}_{\overline{n|}\,i}$ denotes the present value of an annuity immediate that pays $1/m$ at the end of each $m$th part of a period for $n$ periods under the assumption that the *effective* interest rate is $i$ per period. For example, if $m = 12$ and the period is a year, payments of 1/12 are made at the end of each month. What is a formula for $a^{(m)}_{\overline{n|}}$ assuming the effective rate of interest is $i$ per period? Notice here that the payments are made more frequently than interest is compounded. Using the equivalent rate $i^{(m)}$ makes the computations easy. Using geometric series, $a^{(m)}_{\overline{n|}} = \frac{1}{m}\sum_{j=1}^{nm}(1+i^{(m)}/m)^{-j} = (1-v^n)/i^{(m)} = ia_{\overline{n|}}/i^{(m)}$.

**Exercise 4–6.** The symbol $\ddot{a}^{(m)}_{\overline{n|}\,i}$ denotes the present value of an annuity due that pays $1/m$ at the beginning of each $m$th part of a period for $n$ periods when the effective periodic interest rate is $i$. Find a formula for $\ddot{a}^{(m)}_{\overline{n}|}$ assuming the effective periodic rate of interest is $i$.

**Exercise 4–7.** The symbol $(Ia)^{(m)}_{\overline{n}|}$ is the present value of an annuity that pays $1/m$ at the end of each $m$th part of the first period, $2/m$ at the end of each $m$th part of the second period, ..., $n/m$ at the end of each $m$th part of the $n$th period when the effective annual interest rate is $i$. Find a computational formula for $(Ia)^{(m)}_{\overline{n}|}$.

**Exercise 4–8.** The symbol $(I^{(m)}a)^{(m)}_{\overline{n}|}$ is the present value of the annuity which pays $1/m^2$ at the end of the first $m$th of the first period, $2/m^2$ at the end of the second $m$th of the first period,..., $nm/m^2$ at the end of last $m$th of the $n$th period. Give a computational formula for $(I^{(m)}a)^{(m)}_{\overline{n}|}$.

Thus far the value of an annuity has been computed at time 0. Another common time point at which the value of an annuity consisting of $n$ payments of 1 is computed is time $n$. Denote by $s_{\overline{n}|}$ the value of an annuity immediate at time $n$, that is, immediately after the $n$th payment. Then $s_{\overline{n}|} = (1 + i)^n a_{\overline{n}|}$ from the time diagram. The value $s_{\overline{n}|}$ is called the **accumulated value of the annuity immediate**. Similarly $\ddot{s}_{\overline{n}|}$ is the **accumulated value of an annuity due** at time $n$ and $\ddot{s}_{\overline{n}|} = (1 + i)^n \ddot{a}_{\overline{n}|}$.

**Exercise 4–9.** Similarly, $s^{(m)}_{\overline{n}|}$, $\ddot{s}^{(m)}_{\overline{n}|}$, and $\bar{s}_{\overline{n}|}$ are the values of the corresponding annuities just after time $n$. Find a formula for each of these in terms of $s_{\overline{n}|}$.

**Exercise 4–10.** What do the symbols $(Is)_{\overline{n}|}$ and $(I\ddot{s})_{\overline{n}|}$ represent?

Now a common use of annuities will be examined.

**Example 4–5.** You are going to buy a house for which the purchase price is \$100,000 and the downpayment is \$20,000. You will finance the \$80,000 by borrowing this amount from a bank at 10% interest with a 30 year term. What is your monthly payment? Typically such a loan is *amortized*, that is, you will make equal monthly payments for the life of the loan and each payment consists partially of interest and partially of principal. From the banks point of view this transaction represents the purchase by the bank of an annuity immediate. The monthly payment, $p$, is thus the solution of the equation $80000 = pa_{\overline{360}|0.10/12}$. In this setting the quoted interest rate on the loan is assumed to be compounded at the same frequency as the payment period unless stated otherwise.

**Exercise 4–11.** Find the monthly payment. What is the total amount of the payments made?

An **amortization table** is a table which lists the principal and interest portions of each payment for a loan which is being amortized. An amortization table can be constructed from first principles. Denote by $b_k$ the loan balance immediately after the $k$th payment and write $b_0$ for the original loan amount. Then the interest part of the $k$th payment is $ib_{k-1}$ and the principal amount of the $k$th payment is $P - ib_{k-1}$ where $P$ is the periodic payment amount. Notice too that $b_{k+1} = (1+i)b_k - P$. These relations allow the rows of the amortization table to be constructed sequentially.

Taking a more sophisticated viewpoint will exhibit a method of constructing any single row of the amortization table that is desired, without constructing the whole table. In the **prospective method** the loan balance at any point in time is seen to be the present value of the remaining loan payments. The prospective method gives the loan balance immediately after the $k$th payment as $b_k = Pa_{\overline{n-k}|}$. In the **retrospective method** the loan balance at any point in time is seen to be the accumulated original loan amount less the accumulated value of the past loan payments. The retrospective method gives the loan balance immediately after the $k$th payment as $b_k = b_0(1+i)^k - Ps_{\overline{k}|}$. Either method can be used to find the loan balance at an arbitrary time point. With this information, any given row of the amortization table can be constructed.

**Exercise 4–12.** Show that for the retrospective method, $b_0(1+i)^k - Ps_{\overline{k}|} = b_0 + (ib_0 - P)s_{\overline{k}|}$.

**Exercise 4–13.** Show that the prospective and retrospective methods give the same value.

A further bit of insight is obtained by examining the case in which the loan amount is $a_{\overline{n}|}$, so that each loan payment is 1. In this case the interest part of the $k$th payment is $ia_{\overline{n-k+1}|} = 1 - v^{n-k+1}$ and the principal part of the $k$th payment is $v^{n-k+1}$. This shows that the principal payments form a geometric series.

Finally observe that the prospective and retrospective method apply to *any* series of loan payments. The formulas obtained when the payments are not all equal will just be messier.

A second way of paying off a loan is by means of a sinking fund.

**Example 4–6.** As in the previous example, \$80,000 is borrowed at 10% annual interest. But this time, only the interest is required to be paid each month. The principal amount is to be repaid in full at the end of 30 years. Of course, the borrower wants to accumulate a separate fund, called a **sinking fund**, which will accumulate to \$80,000 in 30 years. The borrower can only earn 5% interest compounded monthly. In this scenario, the monthly interest payment is $80000(0.10/12) = 666.67$. The contribution $c$ each month into the sinking fund must satisfy $cs_{\overline{360}|0.05/12} = 80000$, from which $c = 96.12$. As expected, the combined payment is higher, since the interest rate earned on the sinking fund is lower than 10%.

Here is a summary of the most useful formulas thus far.

$$a_{\overline{n}|} = \frac{1-v^n}{i} \qquad s_{\overline{n}|} = \frac{v^{-n}-1}{i} = v^{-n}a_{\overline{n}|}$$

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} \qquad \ddot{s}_{\overline{n}|} = \frac{v^{-n}-1}{d} = v^{-n}\ddot{a}_{\overline{n}|}$$

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|}-nv^n}{i} \qquad (Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|}-n}{i} = v^{-n}(Ia)_{\overline{n}|}$$

$$(Da)_{\overline{n}|} = \frac{n-a_{\overline{n}|}}{i} \qquad (Ds)_{\overline{n}|} = \frac{nv^{-n}-s_{\overline{n}|}}{i} = v^{-n}(Da)_{\overline{n}|}$$

$$v\ddot{a}_{\overline{n}|} = a_{\overline{n}|}$$

The reader should have firmly in mind the time diagram for each of the basic annuities, as well as these computational formulas.

## Problems

**Problem 4–1.** Show that $a_{\overline{n}|} < \bar{a}_{\overline{n}|} < \ddot{a}_{\overline{n}|}$. Hint: This should be obvious from the picture.

**Problem 4–2.** True or False: For any two interest rates $i$ and $i'$, $(1+i)^{-n}(1+i's_{\overline{n}|i}) = 1+(i'-i)a_{\overline{n}|i}$.

**Problem 4–3.** True or False: $\dfrac{1}{a_{\overline{n}|}} = \dfrac{1}{s_{\overline{n}|}} + i$.

**Problem 4–4.** True or False: $\ddot{a}^{(m)}_{\overline{n}|} = \left(\dfrac{i}{i^{(m)}} + \dfrac{i}{m}\right) a_{\overline{n}|}$.

**Problem 4–5.** Show that $a_{\overline{2n}|} = a_{\overline{n}|}(1+v^n)$.

**Problem 4–6.** Show that $a_{\overline{3n}|} = a_{\overline{n}|} + v^n a_{\overline{2n}|} = a_{\overline{2n}|} + v^{2n} a_{\overline{n}|}$.

**Problem 4–7.** Suppose an annuity immediate pays $p$ at the end of the first period, $pr$ at the end of the second period, $pr^2$ at the end of the third period, and so on, until a final payment of $pr^{n-1}$ is made at the end of the $n$th period. What is the present value of this annuity?

**Problem 4–8.** John borrows \$1,000 from Jane at an annual effective rate of interest $i$. He agrees to pay back \$1,000 after six years and \$1,366.87 after another 6 years. Three years after his first payment, John repays the outstanding balance. What is the amount of John's second payment?

**Problem 4–9.** A loan of 10,000 carries an interest rate of 9% compounded quarterly. Equal loan payments are to be made monthly for 36 months. What is the size of each payment?

**Problem 4–10.** An annuity immediate pays an initial benefit of one per year, increasing by 10.25% every four years. The annuity is payable for 40 years. If the effective interest rate is 5% find an expression for the present value of this annuity.

**Problem 4–11.** Humphrey purchases a home with a \$100,000 mortgage. Mortgage payments are to be made monthly for 30 years, with the first payment to be made one month from now. The rate of interest is 10%. After 10 years, Humphrey increases the amount of each monthly payment by \$325 in order to repay the mortgage more quickly. What amount of interest is paid over the life of the loan?

**Problem 4–12.** On January 1, an insurance company has \$100,000 which is due to Linden as a life insurance death benefit. He chooses to receive the benefit annually over a period of 15 years, with the first payment made immediately. The benefit he receives is based on an effective interest rate of 4% per annum. The insurance company earns interest at an effective rate of 5% per annum. Every July 1 the company pays $100 in expenses and taxes to maintain the policy. How much money does the company have remaining after 9 years?

**Problem 4–13.** A loan of 10,000 is to be repaid with equal monthly payments of $p$. The interest rate for the first year is 1.9%, while the interest rate for the remaining 2 years is 10.9%. What is $p$? What is the balance after the 6th payment? After the 15th payment? What are the principal and interest components of the 7th payment? Of the 16th payment?

**Problem 4–14.** A loan of 10,000 is to be repaid as follows. Payments of $p$ are to be made at the end of each month for 36 months and a balloon payment of 2500 is to be made at the end of the 36th month as well. If the interest rate is 5%, what is $p$? What is the loan balance at the end of the 12th month? What part of the 13th payment is interest? Principal?

**Problem 4–15.** A loan is being amortized with a series of 20 annual payments. If the amount of principal in the third payment is 200, what is the amount of principal in the last 3 payments? The interest rate is 4%.

**Problem 4–16.** A loan is amortized with 10 annual installments. The principal part of the fifth payment is 20 and the interest part is 5. What is the rate of interest on the loan?

**Problem 4–17.** A loan is amortized with payments of 1 at the end of each year for 20 years. Along with the fifth payment the borrower sends the amount of principal which would have been paid with the sixth payment. At the time of the sixth payment, the borrower resumes payment of 1 until the loan is repaid. How much interest is saved by this single modified payment?

**Problem 4–18.** A loan of 1000 is being repaid by equal annual installments of 100 together with a smaller final payment at the end of 10 years. If the interest rate is 4%, show that the balance immediately after the fifth payment is $1000 - 60s_{\overline{5}|.04}$.

**Problem 4–19.** A loan of 1200 is to be repaid over 20 years. The borrower is to make annual payments of 100 at the end of each year. The lender receives 5% on the loan for the first 10 years and 6% on the loan balance for the remaining years. After accounting for the interest to be paid, the remainder of the payment of 100 is deposited in a sinking fund earning 3%. What is the loan balance still due at the end of 20 years?

**Problem 4–20.** A loan is being repaid with 10 payments. The first payment is 10, the second payment is 9, and so on. What is the amount of interest in the fourth payment?

**Problem 4–21.** A standard **perpetuity immediate** is an annuity which pays 1 at the end of each period forever. What is $a_{\overline{\infty}|}$, the present value of a standard perpetuity immediate? A standard **perpetuity due** pays 1 at the beginning of each period forever. What is $\ddot{a}_{\overline{\infty}|}$, the present value of a standard perpetuity due?

**Problem 4–22.** A standard perpetuity due has a present value of 20, and will be exchanged for a perpetuity immediate which pays $R$ per period. What is the value of $R$ that makes these two perpetuities of equal value?

**Problem 4–23.** You are given an annuity immediate paying \$10 for 10 years, then decreasing by \$1 per year for nine years and paying \$1 per year thereafter, forever. If the annual effective rate of interest is 5%, find the present value of this annuity.

**Problem 4–24.** A loan is being repaid with a payment of 200 at the end of the first year, 190 at the end of the second year, and so on, until the final payment of 110 at the end of the tenth year. If the interest rate is 6%, what was the original loan amount?

**Problem 4–25.** A loan is being repaid with a payment of 200 at the end of the first year, 190 at the end of the second year, and so on, until the final payment of 110 at the end of the tenth year. The borrower pays interest on the original loan amount at a rate of 7%, and contributes the balance of each payment to a sinking fund that earns 4%. If the amount in the sinking fund at the end of the tenth year is equal to the orginal loan amount, what was the original loan amount?

**Problem 4–26.** A loan of 1 was to be repaid with 25 equal payments at the end of the year. An extra payment of $K$ was made in addition to the sixth through tenth payments, and these extra payments enabled the loan to be repaid five years early. Show that $K = (a_{\overline{20}|} - a_{\overline{15}|})/a_{\overline{25}|}a_{\overline{5}|}$.

**Problem 4–27.** A loan is being repaid with quarterly payments of 500 at the end of each quarter for seven years at an interest rate of 8% compounded quarterly. What is the amount of principal in the fifth payment?

**Problem 4–28.** A borrower is repaying a loan with 20 annual payments of 500 made at the end of each year. Half of the loan is repaid by the amortization method at 6% effective. The other half of the loan is repaid by the sinking fund method in which the interest rate is 6% effective and the sinking fund accumulates at 5% effective. What is the amount of the loan?

**Problem 4–29.** A loan of 18000 is made for 12 years in which the lender receives 6% compounded semiannually for the first six years and 4% compounded semiannually for the last six years. The borrower makes seminannual payments of 1000, and the balance after paying interest is deposited into a sinking fund which pays 3% compounded semiannually. What is the net amount remaining on the loan after 12 years?

**Problem 4–30.** The interest on an inheritance invested at 4% effective would have been just sufficient to pay 16000 at the end of each year for 15 years. Payments were made as planned for the first five years, even though the actual interest earned on the inheritance for years 3 through 5 was 6% effective instead of 4% effective. How much excess interest had accumulated at the end of the fifth year?

## Solutions to Problems

**Problem 4–1.** Isn't the chain of inequalities simply expressing the fact that getting a given amount of money sooner makes it worth more? An analytic proof should be easy to give too.

**Problem 4–2.** True, since $v^n(1 + i' s_{\overline{n}|i}) = v^n + i' a_{\overline{n}|i} = 1 - i a_{\overline{n}|i} + i' a_{\overline{n}|i}$.

**Problem 4–3.** True. Write $1/(1 - v^n) = v^{-n}/(v^{-n} - 1) = (v^{-n} - 1 + 1)/(v^{-n} - 1) = 1 + 1/(v^{-n} - 1)$, multiply by $i$ and use the definitions. This also has a verbal explanation. $1/a_{\overline{n}|}$ is the periodic payment to amortize a loan of 1 with $n$ payments. The loan can also be paid off by paying $i$ per period and contributing $1/s_{\overline{n}|}$ to a sinking fund. Similar reasoning shows that $1/a^{(m)}_{\overline{n}|} = i^{(m)} + 1/s^{(m)}_{\overline{n}|}$ and $1/\ddot{a}^{(m)}_{\overline{n}|} = d^{(m)} + 1/\ddot{s}^{(m)}_{\overline{n}|}$.

**Problem 4–4.** True, since $\ddot{a}^{(m)}_{\overline{n}|} = (1 + i)^{1/m} a^{(m)}_{\overline{n}|} = (1 + i^{(m)}/m)(i/i^{(m)}) a_{\overline{n}|}$.

**Problem 4–5.** Breaking the period of length $2n$ into two periods of length $n$ gives $a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|}$, and the result follows.

**Problem 4–6.** Just break the period of length $3n$ into two pieces, one of length $n$ and the other of length $2n$.

**Problem 4–7.** Direct computation gives the present value as $\sum_{j=1}^{n} p r^{j-1} v^j = (pv - pv^{n+1} r^n)/(1 - vr) = p(1 - v^n r^n)/(1 + i - r)$, provided $vr \neq 1$.

**Problem 4–8.** From Jane's point of view the equation $1000 = 1000(1 + i)^{-6} + 1366.87(1 + i)^{-12}$ must hold. The outstanding balance at the indicated time is $1366.87(1 + i)^{-3}$, which is the amount of the second payment.

**Problem 4–9.** An interest rate of 9% compounded quarterly is equivalent to an interest rate of 8.933% compounded monthly. The monthly payment is therefore $10000/a_{\overline{36}|.0893/12} = 10000/31.47 = 317.69$.

**Problem 4–10.** Each 4 year chunk is a simple annuity immediate. Taking the present value of these chunks forms an annuity due with payments every 4 years that are increasing.

**Problem 4–11.** The initial monthly payment $P$ is the solution of $100,000 = Pa_{\overline{360}|}$. The balance after 10 years is $Pa_{\overline{240}|}$ so the interest paid in the first 10 years is $120P - (100,000 - Pa_{\overline{240}|})$. To determine the number of new monthly payments required to repay the loan the equation $Pa_{\overline{240}|} = (P + 325)a_{\overline{x}|}$ should be solved for $x$. Since after $x$ payments the loan balance is 0 the amount of interest paid in the second stage can then be easily determined.

**Problem 4–12.** Since the effective rate of interest for the insurance company is 5%, the factor $(1.05)^{-1/2}$ should be used to move the insurance company's expenses from July 1 to January 1.

**Problem 4–13.** The present value of the monthly payments must equal the original loan amount. Thus $10000 = pa_{\overline{12}|0.019/12} + (1+0.019/12)^{-12} pa_{\overline{24}|0.109/12}$, from which $p = 303.49$. The balance after the sixth payment is most easily found by the retrospective method. The balance is $10000(1+0.019/12)^6 - 303.49 s_{\overline{6}|0.019/12} = 8267.24$. The balance after the 15th payment is most easily found by the prospective method. The balance is $303.49 a_{\overline{21}|0.109/12} = 5778.44$. The interest portion of the 7th payment is $(0.0109/12)(8267.24) = 13.09$ and the principal portion is $303.49 - 13.09 = 290.40$. Similar computations give the interest portion of the 16th payment as 52.49 and the principal portion as 251.00.

**Problem 4–14.** The payment $p = 235.20$ since $10000 = pa_{\overline{36}|.05/12} + (1 + .05/12)^{-36} 2500$. Using the retrospective method, the loan balance at the end of the 12th month is $10000(1+.05/12)^{12} - ps_{\overline{12}|.05/12} = 7623.65$. The interest part of the 13th payment is 31.77.

**Problem 4–15.** If $p$ is the annual payment amount, the principal part of the third payment is $pv^{18} = 200$. Thus $p = 405.16$, and the loan balance after the seventeenth payment is $pa_{\overline{3}|0.04} = 1124.36$.

**Problem 4–16.** Let $p$ denote the annual payment amount. Then $pv^6 = 20$ and $p(1-v^6) = 5$. Adding these two equations gives $p = 25$, so $v^6 = 0.8$, from which $i = 0.0379$.

**Problem 4–17.** The borrower saves the interest on the sixth payment, which is $1 - v^{20-6+1} = 1 - v^{15}$.

**Problem 4–18.** The retrospective method gives the balance as $1000v^{-5} - 100s_{\overline{5}|.04}$, which re-arranges to the stated quantity using the identity $v^{-n} = 1 + is_{\overline{n}|}$.

**Problem 4–19.** The amount in the sinking fund at the end of 20 years is $40s_{\overline{10}|}(1.03)^{10} + 28s_{\overline{10}|} = 937.25$, so the loan balance is $1200 - 937.25 = 262.75$.

**Problem 4–20.** The loan balance immediately after the third payment is $(Da)_{\overline{7}|} = (7 - a_{\overline{7}|})/i$, so the interest paid with the fourth payment is $7 - a_{\overline{7}|}$.

**Problem 4–21.** Summing the geometric series gives $a_{\overline{\infty}|} = 1/i$ and $\ddot{a}_{\overline{\infty}|} = 1/d$. These results can also be obtained by letting $n \to \infty$ in the formulas for $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$.

**Problem 4–22.** From the information given, $1/d = 20$ (and so $d = 1/20$) and $Rv/d = 20$. Since $v = 1 - d = 19/20$, $R = 20/19$.

**Problem 4–23.** What is the present value of an annuity immediate paying \$1 per year forever? What is the present value of such an annuity that begins payments $k$ years from now? The annuity described here is the difference of a few of these.

**Problem 4–24.** The payments consist of a level payment of 100 together with a decreasing annuity. The present value of these payments at the date of issue of the loan is the loan amount, which is therefore $100a_{\overline{10}|.06} + 10(Da)_{\overline{10}|.06} = 1175.99$.

**Problem 4–25.** Let $A$ be the original loan amount. The contribution to the sinking fund at the end of year $j$ is then $100 + 10(11 - j) - .07A$, and these contributions must accumulate to $A$. Thus $100s_{\overline{10}|.04} + 10(Ds)_{\overline{10}|.04} - .07As_{\overline{10}|.04} = A$, from which $A = 1104.24$.

**Problem 4–26.** Denote the original payment amount by $p$. Then $p = 1/a_{\overline{25}|}$. The fact that the additional payments extinguish the loan after 20 years means $1 = pa_{\overline{20}|} + v^5 K a_{\overline{5}|}$. Thus $K = (a_{\overline{25}|} - a_{\overline{20}|})/v^5 a_{\overline{5}|} a_{\overline{25}|}$. The result follows since $v^{-5}(a_{\overline{25}|} - a_{\overline{20}|}) = a_{\overline{20}|} - a_{\overline{15}|}$.

**Problem 4–27.** Using $i = 0.02$, the loan balance immediately after the fourth payment is $500a_{\overline{24}|}$, so the principal part of the fifth payment is $500 - 10a_{\overline{24}|} = 310.86$.

**Problem 4–28.** Denote by $L$ the loan amount. From the information given, $(500 - L/(2a_{\overline{20}|0.06}) - 0.03L)s_{\overline{20}|0.05} = L/2$, so that $L = 500/(1/2s_{\overline{20}|0.05} + 0.03 + 1/2a_{\overline{20}|0.06}) = 5636.12$.

**Problem 4–29.** The semiannual contributions to the sinking fund are 460 for the first six years and 640 for the last six years. The sinking fund balance at the end of 12 years is $(1.015)^{12} 460 s_{\overline{12}|0.015} + 640 s_{\overline{12}|0.015} = 15518.83$. So the net loan balance is $18000 - 15518.83 = 2481.17$.

**Problem 4–30.** The accumulated interest is $(24000 - 16000)s_{\overline{3}|.06} = 25468.80$.

## Solutions to Exercises

**Exercise 4–1.** The sum is the sum of the terms of a geometric series. So $\sum_{j=1}^{240}(1+\frac{.05}{12})^{-j}\,400 = 400((1+0.05/12)^{-1}-(1+0.05/12)^{-241})/(1-(1+0.05/12)^{-1}) = 60{,}610.12$.

**Exercise 4–2.** This follows from the formulas for the present value of the two annuities and the fact that $d = iv$.

**Exercise 4–3.** The method of the example could be used again, but the formula can also be obtained from $(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} = (n+1)a_{\overline{n}|}$, which gives $(Da)_{\overline{n}|} = (n - a_{\overline{n}|})/i$.

**Exercise 4–4.** The present value is $(Ia)_{\overline{n}|} + v^n(Da)_{\overline{n-1}|} = (1 + a_{\overline{n-1}|} - v^n - v^n a_{\overline{n-1}|})/i = (1-v^n)(1 + a_{\overline{n-1}|})/i = \ddot{a}_{\overline{n}|}a_{\overline{n}|}$.

**Exercise 4–5.** This is the present value of an annuity with payments of 1 which start at the beginning of period $k+1$ (that is, at time $k$) and continue for a total of $n$ payments. Thus ${}_{k|}\ddot{a}_{\overline{n}|} = v^k\ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{k+n}|} - \ddot{a}_{\overline{k}|}$.

**Exercise 4–6.** As in the previous example, $\ddot{a}^{(m)}_{\overline{n}|} = (1-v^n)/d^{(m)} = d\ddot{a}_{\overline{n}|}/d^{(m)}$.

**Exercise 4–7.** Proceeding as in the derivation of the formula for $(Ia)_{\overline{n}|}$ gives $(Ia)^{(m)}_{\overline{n}|} = \dfrac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}}$.

**Exercise 4–8.** $(I^{(m)}a)^{(m)}_{\overline{n}|} = \dfrac{\ddot{a}^{(m)}_{\overline{n}|} - nv^n}{i^{(m)}}$ using the same techniques as before.

**Exercise 4–9.** Direct computation using the parallel facts for the values at time 0 give $s^{(m)}_{\overline{n}|} = is_{\overline{n}|}/i^{(m)}$, $\ddot{s}^{(m)}_{\overline{n}|} = is_{\overline{n}|}/d^{(m)}$, and $\bar{s}_{\overline{n}|} = is_{\overline{n}|}/\delta$.

**Exercise 4–10.** The symbol $(Is)_{\overline{n}|}$ is the value of an increasing annuity immediate computed at time $n$; $(I\ddot{s})_{\overline{n}|}$ is the value of an increasing annuity due at time $n$.

**Exercise 4–11.** Using the earlier formula gives $a_{\overline{360}|0.10/12} = 113.95$ from which $p = 702.06$ and the total amount of the payments is $360p = 252740.60$.

**Exercise 4–12.** Simply use the identity $v^{-n} = 1 + is_{\overline{n}|}$.

**Exercise 4–13.** Direct computation using the formulas gives $a_{\overline{n-k}|} = v^{-k}(a_{\overline{n}|} - a_{\overline{k}|})$ and $P = b_0/a_{\overline{n}|}$ gives $Pa_{\overline{n-k}|} = b_0 v^{-k}(a_{\overline{n}|} - a_{\overline{k}|})/a_{\overline{n}|} = b_0(1+i)^k - Ps_{\overline{k}|}$.

## §5. Sample Question Set 2

**Solve the following 13 problems in no more than 65 minutes.**

**Question 5–1 .** The present value of a series of payments of 2 at the end of every eight years, forever, is equal to 5. Calculate the effective rate of interest.

**A.** 0.023

**B.** 0.033

**C.** 0.040

**D.** 0.043

**E.** 0.052

**Question 5–2 .** An annuity immediate pays an initial benefit of one per year, increasing by 10.25% every four years. The annuity is payable for 40 years. Using an annual effective interest rate of 5%, determine an expression for the present value of this annuity.

**A.** $(1 + v^2)\ddot{a}_{\overline{20}|}$

**B.** $(1 + v^2)a_{\overline{20}|}$

**C.** $2\,a_{\overline{20}|}$

**D.** $\dfrac{a_{\overline{20}|}}{s_{\overline{2}|}}$

**E.** $\dfrac{a_{\overline{40}|}}{a_{\overline{2}|}}$

**Question 5–3 .** Determine an expression for $\dfrac{a_{\overline{5}|}}{a_{\overline{6}|}}$.

**A.** $\dfrac{a_{\overline{2}|} + a_{\overline{3}|}}{2\,a_{\overline{3}|}}$

**B.** $\dfrac{a_{\overline{2}|} + s_{\overline{3}|}}{1 + a_{\overline{3}|} + s_{\overline{2}|}}$

**C.** $\dfrac{a_{\overline{2}|} + s_{\overline{3}|}}{a_{\overline{3}|} + s_{\overline{3}|}}$

**D.** $\dfrac{1 + a_{\overline{2}|} + s_{\overline{2}|}}{a_{\overline{3}|} + s_{\overline{3}|}}$

**E.** $\dfrac{1 + a_{\overline{2}|} + s_{\overline{2}|}}{1 + a_{\overline{3}|} + s_{\overline{2}|}}$

**Question 5–4 .** Which of the following are true?

I. $(\bar{a}_{\overline{n|}} - \frac{d}{\delta})(1+i) = \bar{a}_{\overline{n-1|}}$

II. The present value of a 10 year temporary annuity immediate paying 10 per month for the first eight months of each year is $120 a_{\overline{10|}}\, a^{(12)}_{\overline{8/12|}}$.

III. The present value of a perpetuity paying one at the end of each year, except paying nothing every fourth year, is $\frac{s_{\overline{3|}}}{i\, s_{\overline{4|}}}$.

**A.** I and II only

**B.** I and III only

**C.** II and III only

**D.** I, II, and III

**E.** The correct answer is not given by A, B, C, or D

**Question 5–5 .** Warren has a loan with an effective interest rate of 5% per annum. He makes payments at the end of each year for 10 years. The first payment is 200, and each subsequent payment increases by 10 per year. Calculate the interest portion in the fifth payment.

**A.** 58

**B.** 60

**C.** 62

**D.** 65

**E.** 67

**Question 5–6 .** You are given a 15 year mortgage with monthly payments of 1,000 and interest compounded monthly. At the end of each month you make a payment of 1,000. In addition to the regular monthly payment of 1,000, you make an additional payment equal to the amount of principal that would have been repaid in the next regular monthly payment. Under this scheme the loan will be completely repaid after 90 payments. Determine an expression for the amount of interest saved over the life of the loan.

**A.** $1,000(90 - (1+i)\frac{a_{\overline{180|}}}{a_{\overline{2|}}})$

**B.** $1,000(90 - (1+i)\frac{a_{\overline{180|}}}{\ddot{a}_{\overline{2|}}})$

**C.** $1,000(90 - (1+i)\frac{a_{\overline{180|}}}{a_{\overline{3|}}})$

**D.** $1,000(90 - (1+i)\frac{a_{\overline{180|}}}{s_{\overline{2|}}})$

**E.** $1,000(90 - (1+i)\frac{a_{\overline{180|}}}{\ddot{s}_{\overline{2|}}})$

**Question 5–7** . An investment fund accrues interest with force of interest $\delta_t = \frac{K}{1+(1-t)K}$ for $0 \le t \le 1$. At time zero, there is 100,000 in the fund. At time one there is 110,000 in the fund. The only two transactions during the year are a deposit of 15,000 at time 0.25 and a withdrawal of 20,000 at time 0.75. Calculate $K$.

**A.** 0.047

**B.** 0.051

**C.** 0.141

**D.** 0.150

**E.** 0.154

**Question 5–8** . You are given an annuity immediate with 11 annual payments of 100 and a final balloon payment at the end of 12 years. At an annual effective interest rate of 3.5%, the present value at time 0 of all the payments is 1,000. Using an annual effective interest rate of 1%, calculate the present value at the beginning of the ninth year of all remaining payments.

**A.** 412

**B.** 419

**C.** 432

**D.** 439

**E.** 446

**Question 5–9** . Using an annual effective interest rate $j > 0$, you are given

(1) The present value of 2 at the end of each year for $2n$ years, plus an additional 1 at the end of each of the first $n$ years, is 36

(2) The present value of an $n$-year deferred annuity immediate paying 2 per year for $n$ years is 6

Calculate $j$.

**A.** 0.03

**B.** 0.04

**C.** 0.05

**D.** 0.06

**E.** 0.07

**Question 5–10** . An 11 year annuity has a series of payments 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1, with the first payment made at the end of the second year. The present value of this annuity is 25 at interest rate $i$. A 12 year annuity has a series of payments 1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 1, with the first payment made at the end of the first year. Calculate the present value of the 12 year annuity at interest rate $i$.

**A.** 29.5

**B.** 30.0

**C.** 30.5

**D.** 31.0

**E.** 31.5

**Question 5–11** . Joan has won a lottery that pays 1,000 per month in the first year, 1,100 per month in the second year, 1,200 per month in the third year, etc. Payments are made at the end of each month for 10 years. Using an effective interest rate of 3% per annum, calculate the present value of this prize.

**A.** 107,000

**B.** 114,000

**C.** 123,000

**D.** 135,000

**E.** 148,000

**Question 5–12** . A 5% 10 year loan of 10,000 is to be repaid by the sinking fund method, with interest and sinking fund payments made at the end of each year. The effective rate of interest earned in the sinking fund is 3% per annum. Immediately before the fifth year's payment would have fallen due, the lender requests that the outstanding principal be repaid in one lump sum. Calculate the amount that must be paid, including interest, to extinguish the debt.

**A.** 6,350

**B.** 6,460

**C.** 6,740

**D.** 6,850

**E.** 7,000

**Question 5–13** . A company agrees to repay a loan over five years. Interest payments are made annually and a sinking fund is built up with five equal annual payments made at the end of each year. Interest on the sinking fund is compounded annually. You are given

(1) The amount in the sinking fund immediately after the first payment is $X$

(2) The amount in the sinking fund immediately after the second payment is $Y$

(3) $Y/X = 2.09$

(4) The net amount of the loan immediately after the fourth payment is 3,007.87

Calculate the amount of the sinking fund payment.

**A.** 1,931

**B.** 2,031

**C.** 2,131

**D.** 2,231

**E.** 2,431

## Answers to Sample Questions

**Question 5–1** . The information given is that $2\sum_{j=1}^{\infty} v^{8j} = 5$. Using geometric series gives this equation as $v^8/(1-v^8) = 5/2$, from which $v = (5/7)^{1/8}$ and $i = 0.0429$. **D** .

**Question 5–2** . Note that $(1.05)^2 = 1.1025$, so that the quadrennial increase is $v^{-2}$. Writing out directly the present value of the payments gives $(v + v^2 + v^3 + v^4) + (v^3 + v^4 + v^5 + v^6) + (v^5 + v^6 + v^7 + v^8) + \dots + (v^{19} + v^{20} + v^{21} + v^{22})$ which re-arranges to $a_{\overline{20}|} + v^2 a_{\overline{20}|}$, or **B**.

**Question 5–3** . Write $a_{\overline{5}|} = v + v^2 + v^3 + v^4 + v^5$, and to likewise for $a_{\overline{6}|}$. Multiply top and bottom by $v^{-3}$ to get answer **C**.

**Question 5–4** . That I is true follows from $(\bar{a}_{\overline{n}|} - d/\delta)(1 + i) = (1 - d - e^{-n\delta})e^{\delta}/\delta = (ve^{\delta} - e^{(n-1)\delta})/\delta = \bar{a}_{\overline{n-1}|}$. II is false, since the middle factor should be $\ddot{a}_{\overline{10}|}$. III is also false, the given answer should be multiplied by $v^3$ to be correct. **E**.

**Question 5–5** . The loan balance at the beginning of the fifth year is $240v + 250v^2 + \dots + 290v^6 = 1337.84$, where $v = 1/(1.05)$. The interest for the fifth year is therefore $0.05(1337.84) = 66.89$. **E**.

**Question 5–6** . The new scheme results in the borrower skipping all of the even numbered interest payments in the original loan. The interest saved is $1000\sum_{j=1}^{90}(1 - v^{181-2j}) = 1000(90 - v\frac{1-v^{180}}{1-v^2}) = 1000(90 - (1 + i)\frac{a_{\overline{180}|}}{s_{\overline{2}|}})$. **D**.

**Question 5–7** . The equation of value is $100e^{\int_0^1 \delta_t\,dt} + 15e^{\int_{1/4}^1 \delta_t\,dt} - 20e^{\int_{3/4}^1 \delta_t\,dt} = 110$. Simplifying gives $(425/4)K + 95 = 110$ from which $K = 12/85 = 0.1411$. **C**.

**Question 5–8** . From the given information, $1000 = 100a_{\overline{11}|.035} + (1.035)^{-12}B$, where $B$ is the amount of the balloon payment. Thus $B = 150.87$. The present value at the beginning of the ninth year is $100a_{\overline{3}|.01} + (1.01)^{-4}B = 439.08$. **D**.

**Question 5–9** . Looking at the first annuity as paying 3 for $2n$ years and taking back 1 in each of the last $n$ years shows that $36 = 3a_{\overline{2n}|} - 6/2$, from which $a_{\overline{2n}|} = 13$. Similarly, $6 = 2a_{\overline{2n}|} - 2a_{\overline{n}|}$, so $a_{\overline{n}|} = 10$. But $6 = 2v^n a_{\overline{n}|}$ also, so that $v^n = 6/20$ Finally, $10 = a_{\overline{n}|} = (1 - v^n)/i = (1 - (6/20))/i$, from which $i = 7/100$. **E**.

**Question 5–10** . Write $T$ for the present value of the twelve year annuity and $E$ for the present value of the eleven year annuity. Looking at a time diagram shows that $T - E = a_{\overline{6}|}$, and direct computation gives $E = v\ddot{a}_{\overline{6}|}a_{\overline{6}|} = (a_{\overline{6}|})^2$. Thus $T = E + a_{\overline{6}|} = 25 + 5 = 30$. **B**.

**Question 5–11** . By breaking off the increasing part from a constant payment of

1000 per month for the life of the payments, the present value is $1000 s_{\overline{12}|(.03)^{(12)}} a_{\overline{10}|.03} + (1.03)^{-1} 100 s_{\overline{12}|(.03)^{(12)}} (Ia)_{\overline{9}|} = 147928.85$. **E**.

**Question 5–12** . The annual sinking fund contribution is $10000/s_{\overline{10}|.03} = 872.31$. The amount in the sinking fund just before the fifth payment is $872.31 s_{\overline{4}|.03}(1.03) = 3758.88$. The amount due is $10000 - 3758.88 + 500 = 6741.12$. **C**.

**Question 5–13** . The sinking fund payment is $X$. Let the loan amount be $A$. Then $X = A/s_{\overline{5}|}$. Also $Y = X(1+i) + X = (2+i)X = 2.09X$. Thus $i = .09$. Finally $A - Xs_{\overline{4}|} = 3007.87$. So $X = 3007.87/(s_{\overline{5}|} - s_{\overline{4}|}) = 2130.85$. **C**.

## §6. Bonds and Yield Rates

How should two different investment alternatives be compared? The approach commonly taken to answering this question is to compute the yield rate for the two investment alternatives. Intuitively, the yield rate represents the annual rate of return provided by an investment alternative. At least in the case of simple investments, this intuition is realized.

Perhaps the simplest way for a company to raise money is for the company to sell bonds. A **bond** is a promise by the company to repay the original indebtedness represented by the bond, called the **face value** or **par value** of the bond, at a given time in the future, called the **maturity date** of the bond. Normally, the company also promises to pay interest periodically at a prescribed rate, known as the **coupon rate**. The coupon rate is applied to the par value of the bond in order to determine the actual coupon payment. Typically payments are made on a semi-annual basis, so that the coupon rate is a nominal annual rate compounded semi-annually.

**Example 6–1.** The Auburn Football Company needs to raise money to enlarge its production facilities. The company will sell $10,000,000 of bonds. Each bond will have a face value of $1000 and will pay semi-annual interest at the rate of 5%. The bonds will be due 10 years after issue. The owner of one such bond will receive $25 every six months for a total of 20 payments, together with a payment of $1000 10 years after the date of issue.

The example points out that the purchaser of a bond is effectively buying an annuity together with a lump sum payment. If the purchaser is using the coupon rate to value the bond, the value will be exactly the face value of the bond. Otherwise, the price the purchaser is willing to pay will be either less or more than the face value.

**Example 6–2.** If the purchaser wishes to receive a 6% return, compounded semi-annually then the value at issue is $25a_{\overline{20}|0.03} + (1+.03)^{-20}1000 = 925.61$. A purchaser content with a return of 4% compounded semi-annually would value the bond at $25a_{\overline{20}|0.02} + (1 + .02)^{-20}1000 = 1081.76$.

**Example 6–3.** Notice that while the formulas of the preceding example use direct reasoning, alternate expressions are possible. Recalling that $1 - ia_{\overline{n}|i} = v^n$ the price at 6% yield can also be written $1000 + (25 - 30)a_{\overline{20}|0.03}$. This shows directly that if the yield rate is higher than the coupon rate the bond must sell at a discount; a yield rate lower than the coupon rate will cause the bond to sell at a premium. This is also a simpler formula to use computationally since only the annuity value need be computed.

To formulate the notion of yield rate precisely, examine the last example in a slightly different way. The purchaser of a bond is making a cash outlay at time 0 (a negative cash inflow) and will be receiving a positive cash inflow at certain future times. The **yield rate** for this cash flow is the interest rate at which the present value of all of these cash flows (both positive and negative) is zero. In the case of a bond, the price of the bond and the yield rate are related as described in the previous example. The yield rate is also called the **internal rate of return** of the investment. Notice that the concept of yield rate applies to the cash flow associated with any type of investment.

**Example 6–4.** Before the wide availability of calculator's the **bond salesman's method** was used to compute the approximate yield rate of a bond. The bond salesman's method gives the annual yield rate as approximately

$$\frac{\text{Total Interest Paid}+\text{Capital Gain/Loss}}{\text{Average Investment Amount}\times\text{Number of Years}}.$$

**Exercise 6–1.** Suppose a 10000 par bond has 6% semi-annual coupons and matures in 10 years. What is the yield rate if the price is 9000? Compute both exactly and using the bond salesman's method.

Some bonds are callable. A **callable** bond gives the borrower the option of redeeming the bond prior to the maturity date. Usually the borrower must redeem the bond at a higher price than the price at maturity when exercising this option. When valuing a callable bond the investor should assume that the borrower will exercise the option to call the bond to the disadvantage of the investor and compute accordingly.

**Example 6–5.** Suppose a 1000 par value 6% bond has semiannual coupons. Suppose also that the bond is callable at 1050 for the five year period beginning five years after issue and is callable at 1025 thereafter until maturity 15 years after issue. What price should an investor content with a 4% yield pay for the bond? The price should be the smaller of the numbers $1050 + (60 - 40/1.05)a_{\overline{n}|0.02}$ for $10 \le n \le 19$, $1025 + (60 - 40/1.025)a_{\overline{n}|0.02}$ for $20 \le n \le 29$ and $1000 + (60 - 40)a_{\overline{30}|0.02}$. In each case the smallest value occurs when $n$ is the smallest value in the range, so the price is computed to be 1246.76, corresponding to the first option.

**Exercise 6–2.** For the same bond, what price would an investor seeking a 7% yield pay?

**Example 6–6.** Another wrinkle involves short term U.S. government obligations called **Treasury bills**. Treasury bills are government obligations maturing in one year or less. The yield rates are always quoted as a rate of simple discount, and the time period in years is computed using the actual number of days divided by

360. Thus a 13 week treasury bill with a face amount of 10000 which is purchased for 9800 would be quoted as yielding $d$ where $10000(1 - (91/360)d) = 9800$, or $d = 0.0791$.

Especially in the case of corporate bonds, the bonds are traded after issue in much the same way that stocks are traded. The same principles as above relate the yield rate and bond price at any time point before maturity.

**Example 6–7.** Three years after issue the Auburn Football bonds are priced to yield 7%. The price per 1000 bond is therefore $25a_{\overline{14}|0.035} + 1000(1.035)^{-14} = 890.79$.

Now a purchaser of a bond at this price bought the bond at a **discount**; if the price had been more than the redemption value, the purchase would have been made at a **premium**. In the present case, the purchaser will eventually reap a gain of \$109.21 in appreciation in the value of the bond. This fact is reflected for accounting purposes by *marking up* the value of the bond; a bond bought at premium would be *marked down* in a similar fashion. The marking up or marking down is accomplished by apportioning each subsequent coupon payment between interest and principal, in much the same way that a loan is amortized.

To carry out this mark up or mark down a value, called the **book value** of the bond is computed *immediately after* each coupon payment. The book value is simply the present value of the remaining coupon payments and redemption value computed using the yield rate at which the bond was purchased. The interest part of the current coupon is simply the book value at the *previous* time point multiplied by the yield rate; the mark up or mark down is the remainder of the coupon.

**Example 6–8.** The bond of the previous example has an initial book value equal to the purchase price of 890.79. The semi-annual coupon is 25, so the interest part of the first coupon after purchase is $890.79(.035) = 31.18$ and the mark up is $31.18-25 = 6.18$. The book value after the payment of the coupon is $890.79+6.18 = 896.97$. The computations show that the interest payment is larger than the coupon. The appreciation in the book value of the bond is all attributable to interest to the bond holder. This is to be expected, since the bond, being purchased at discount, must produce a yield rate higher than the coupon rate. Writing the purchase price as $890.79 = 1000 + (25 - 35)a_{\overline{14}|.035}$ shows the discount at purchase can be viewed as a loan which is amortized by payments of $25 - 35 = -10$. The principal part of each loan payment is then the amount of markup. By the earlier discussion, the amount of principal in the first payment is $10v^{14+1-1} = 10(1.035)^{-14} = 6.18$. This computation is considerably simpler than the direct approach.

**Example 6–9.** If the same bond had been purchased to yield 3%, the purchase price would have been 1125.43. In this case, the interest part of the first coupon after purchase is $1125.43(.015) = 16.88$ and the mark down is $25 - 16.88 = 8.12$. The book value after the payment of the coupon is 1125.43 – 8.12 = 1117.31. In this case, the excess of the coupon over the interest is a return of the buyer's capital.

The book value can also be computed at times other than immediately after a coupon payment. In order to do this, consider the price that a purchaser would pay for the bond a fractional time $t$ through a coupon period. Assume that the purchaser will obtain a yield rate equal to that of the current bond holder. The purchaser will receive all of the next coupon. The current holder would expect to receive part of this coupon as interest for the period. How should the purchase price be allocated between **accrued interest** (or **accrued coupon**) and price for the bond? The purchase price for the bond is called the **flat price**. The price for the bond is the book value, which is also called (rather misleadingly) the **market price**. The market price is the price commonly quoted in the financial press, since the market price changes smoothly through time, while the flat price fluctuates due to coupon accrual. There are three methods.

The **theoretical method** argues that the flat price should be the book value $B$ after the preceding coupon, accumlated by $(1+i)^t$, where $i$ is the yield rate, giving the flat price of $B(1+i)^t$. The accrued coupon is the coupon amount $c$ accumulated by $s_{\overline{t}|}$, giving $c((1+i)^t-1)/i$. The book value is then the difference $B(1+i)^t-c((1+i)^t-1)/i$.

The **practical method** (useful before calculators became widely available) argues that the flat price is the book value $B$ after the preceding coupon accumulated at simple interest, giving $B(1+it)$. The accumulated coupon is $tc$, and the book value is $B(1+it)-tc$.

The **semi-theoretical method** is the most widely used method, and has been accepted as the standard method of calculation by the securities industry. The flat price is determined as in the theoretical method, and the accrued coupon is determined as in the practical method.

**Example 6–10.** As mentioned earlier, yield rates are not necessarily unique. Suppose payments of 100 now and 109.20 two years from now are to be made in return for receiving 209 one year from now. The yield rate $i$ then satisfies $100(1+i)^2+109.20=209(1+i)$ from which $i=.04$ or $i=.05$.

The situation in the example is not particularly unusual. A theorem can be proved which states that if the accumulated value of the investment at each time $t$ is always of the same sign (except, of course, at the time of the final payment or payout) then the yield rate is unique. Notice that in the example the accumulated value at time 1 is $-100(1+i)+209>0$ while the value at time zero is negative.

**Example 6–11.** An $n$ year annuity is purchased. At an interest rate of 5% the present value of the annuity is 1000. The purchase price $P$ will allow the buyer to recover the purchase price by investing part of each payment in a sinking fund which pays 4% interest. The yield rate will be 6%. What is the purchase price? Each annuity payment is $1000/a_{\overline{n}|.05}$, and $.06P$ of each payment must be pure income to produce the desired yield rate. The sinking fund deposit is $1000/a_{\overline{n}|.05} - .06P$, and these deposits must accumulate to $P$. Thus $(1000/a_{\overline{n}|.05} - .06P)s_{\overline{n}|.04} = P$, from which $P = 1000s_{\overline{n}|.04}/a_{\overline{n}|.05}(1 + .06s_{\overline{n}|.04})$.

**Example 6–12.** Payments of 500 are invested at the end of each year for 10 years. The payments earn interest at a rate of 5%, but this interest can only be invested to earn 4%. The amount in the fund at the end of the 10 year period is $500(10) + 25(Is)_{\overline{9}|.04} = 6253.82$. Suppose an investor wanted to buy this cash stream to yield 7%. The purchase price would be $(1.07)^{-10}6253.82 = 3179.12$. Notice that in this example the yield rate is used only on the final accumulated value. The hidden assumption here is that none of the intermediate amounts can be touched by the investor, so that only the final accumulated value is relevent.

Determining the yield rate for a fund for which deposits and withdrawals are made at various time points is an important problem. For simplicity, consider the case of a one year time period for which an amount $A$ is on deposit at the beginning of the year and contributions $\{C_t\}$ are made at various time points throughout the year. The contributions may be either positive or negative, with negative values corresponding to withdrawals from the fund. If $i$ is the yield rate for the fund, then the contribution at time $t$ will accumulate to $C_t(1 + i)^{1-t}$ by time 1. If $I$ is the total income earned by the fund during the year, then equating values at the end of the year gives

$$Av^{-1} + \sum_t C_t v^{1-t} = A + \sum_t C_t + I.$$

Maclaurin expansion gives the approximation $(1 + i)^{1-t} \approx 1 + (1-t)i$, and using this approximation leads to $I \approx iA + \sum_t C_t(1-t)i$, from which $i \approx I/(A + \sum_t C_t(1-t))$. Computations are even simpler if the total contribution $C = \sum_t C_t$ can be assumed to have been made at a single time point $t$. If $B$ is the amount in the fund at the end of the year, then $C = B - A - I$ and following the reasoning above gives $i \approx I/(tA + (1-t)B - (1-t)I)$. This formula is most often used when $t = 1/2$.

**Example 6–13.** At the beginning of the year a fund has deposits of 10,000. A deposit of 1000 is made after 3 months and a withdrawal of 2000 is made after 9 months. The amount in the fund at the end of the year is 9500. Using the exact formula gives a yield rate $i$ as the solution of $500 = 10000i + 1000((1 + i)^{3/4} - 1) - 2000((1 + i)^{1/4} - 1)$, from which $i = 0.0487$. Using the first approximation gives $i = 500/(10000 + 1000(3/4) - 2000(1/4)) = 0.0487$ to 3 decimal places. Assuming all of the contributions occured half way through the year (which isn't a very good assumption here) gives $i = 500/(10000/2 + 9500/2 - 500/2) = 0.0526$.

All of these methods are referred to as **dollar weighted rates of interest**.

Many other investment vehicles are available. **Common stock** represents an ownership interest in the company issuing the stock. Theoretically, the value of a share of common stock is the present value, at the yield rate, of the future dividend stream provided by the stock. Since the size of future dividends are unknowable, this present value is computed under assumptions about the dividend stream.

**Example 6–14.** Suppose a stock currently pays a dividend of 1 and the dividend is assumed to increase by 5% each year forever. The price to yield 10% is then $\sum_{j=1}^{\infty}(1.05)^{j-1}/(1.10)^j = 20$.

**Preferred stock** is a form of stock which often carries special voting priviledges or dividends. **Money market funds** operate in much the same way as bank deposits, except that there are often restrictions on the size and frequency of withdrawals and deposits. **Certificates of deposit** are interest bearing intruments issued by banks, usually with fixed interest rates. **Guaranteed investment contracts** are investment intruments issued by insurance companies. **Mutual funds** are pooled investment accounts in which the proceeds are usually invested in stocks or bonds. **Mortgage backed securities** are investment vehicles tied to a pool of mortgages. **Collateralized mortgage obligations** are a more elaborately structured form of mortgage backed securities. **Options** are instruments that give the owner the right to buy or sell another security at some future time and at a specified price. **Futures** are agreements to buy or sell a specified item at a specified future date. Futures are standardized contracts that are traded on exchanges. **Forwards** are futures that are not standardized.

Selling short is a technique that is often used in connection with common stocks when an investor believes that the price of the stock will decline. Futures can also be sold short. The broker arranges for the investor to borrower stock (or futures) and then to sell the shares. Eventually the investor purchases shares to return to the lender. Technically, the yield rate on a short sale does not exist. This is because, as described, the investor makes a profit or loss on an investment of zero. In actuality, regulations require the seller to deposit a **margin** or percentage of the sale price with the broker. The margin remains with the broker until the stock is purchased. The broker pays interest on the margin deposit. The investor must pay any dividends on the stock that fall due during the time until repurchase. Also, the proceeds of the short sale are deposited in a non-interest bearing account until the short position is closed out.

**Example 6–15.** Suppose 100 shares are sold short at a price of 20 per share. The margin is 40%, and interest on the margin account is 4%. The stock is repurchased

1 year later at 15. Thus 2000(.40) = 800 is the required margin deposit, while the gain on the actual stock is 2000 – 1500. The yield rate then satisfies $1 + i = (2000(0.40)(1.04) + 2000 - 1500)/800$, from which $i = 0.665$.

## Problems

**Problem 6–1.** A bond with redemption value of 1000 and coupon rate of 6% payable semi-annually for 20 years is sold 5 years after issue at a price to yield 4%. What price was paid for the bond?

**Problem 6–2.** For the bond of the previous problem, what part of the fourth coupon after the sale is interest? What part is mark down?

**Problem 6–3.** A 26 week treasury bill is bought for 9600 at issue and will mature for 10000. What is the yield rate as quoted using the convention for treasury bills? What is the yield rate computed normally?

**Problem 6–4.** For a bond with par value 1 the coupon rate is 150% of the yield rate and the premium for the bond is $p$. For another bond with par value 1 and the same number of coupons and the same yield rate the coupon rate is 75% of the yield rate. What is the price of the second bond in terms of $p$?

**Problem 6–5.** A 15 year 1000 par bond has 7% semiannual coupons and is callable at par after 10 years. What is the price of the bond to yield 5%?

**Problem 6–6.** A 15 year 1000 par bond has 7% semiannual coupons and is callable at par after 10 years. What is the price of the bond to yield 10%? If the bond were called after 10 years, what would the yield rate be?

**Problem 6–7.** A common stock pays annual dividends at the end of each year. The earnings per share for the most recent year were 8 and are assumed to grow at a rate of 10% per year, forever. The dividend will be 0% of earnings for each of the next 10 years, and 50% of earnings thereafter. What is the theoretical price of the stock to yield 12%?

**Problem 6–8.** A loan of 1000 is being repaid by 10 equal installments of 100 at the end of each year. The loan bears an interest rate of 6%, and interest is also paid at the end each year. At what price would the loan be sold at inception in order to yield the buyer 4%?

**Problem 6–9.** An annuity paying 500 at the end of each year for 10 years is to be purchased to yield 6%. The purchase price can be recaptured by making annual contributions to a sinking fund which pays 5% for each of the first 5 years and 4% for each of the last 5 years. What is the purchase price?

**Problem 6–10.** A used car can be purchased for 5000 cash, or with 1500 down and payments of 2000 at the end of each year for the next two years. If the buyer's interest rate is 10%, which option is chosen?

**Problem 6–11.** A payment of 30000 is made at the end of each year for 20 years to completely repay a loan of 320000. Part of each payment is made into a sinking fund that earns 4% effective which will be used to repay the principal at the end of the period. The remainder is used to pay interest on the loan. What is the effective yield to the lender on this loan?

**Problem 6–12.** A borrower needs 800 to make a payment that is due. The debt can be postponed for a year, but in this case the amount due will be 900. Alternately, the borrower can borrow 1000 from another source and repay this new lender 1100 in one year. If the borrower's interest rate is 9%, which option is chosen?

**Problem 6–13.** A 10% 30 year mortgage requires annual payments of $R$. After making 15 payments the loan is renegotiated to be paid off in 5 more years. The lender will have a yield rate of 9% over the entire life of the loan. What is the revised annual payment?

## Solutions to Problems

**Problem 6–1.** The purchase price is $30a_{\overline{30}|.02} + (1.02)^{-30}1000 = 1223.96$.

**Problem 6–2.** The book value just after the third coupon is $30a_{\overline{27}|.02} + (1.02)^{-27}1000 = 1207.07$. The interest part of coupon four is $1207.07(.02) = 24.14$, and the mark down is $30 - 24.14 = 5.86$.

**Problem 6–3.** The yield rate is the solution $d$ of $10000(1-(182/360)d) = 9600$ giving $d = 0.0791$, while the normal yield rate satisfies $9600(1+i)^{1/2} = 10000$ giving $i = 0.0850$.

**Problem 6–4.** For the first bond $1 + p = 1 + (1.5i - i)a_{\overline{n}|i}$ while for the price $X$ of the second bond $X = 1 + (0.75i - i)a_{\overline{n}|i}$. Solving gives $X = 1 - p/2$.

**Problem 6–5.** The price is the smallest of $1000+(35-25)a_{\overline{n}|0.025}$ for $20 \le n \le 30$, which clearly occurs when $n = 20$ and gives the price as 1155.89.

**Problem 6–6.** The price is $1000+(35-50)a_{\overline{30}|0.05} = 769.41$. Under the alternate conditions the yield rate would be 0.1083.

**Problem 6–7.** The present value of future dividends is $\sum_{k=11}^{\infty} 4(1.1)^k(1.12)^{-k} = 183.73$.

**Problem 6–8.** The payment at the end of year $j$ is $100 + 6(11 - j)$, so the price to yield 4% is $100a_{\overline{10}|.04} + 6(Da)_{\overline{10}|.04} = 1094.46$.

**Problem 6–9.** If the purchase price is $P$, then $P = (500 - .06P)s_{\overline{5}|.05}(1.04)^5 + (500 - .06P)s_{\overline{5}|.04}$, from which $P = 3511.77$.

**Problem 6–10.** The present value of the payments under the second option is $1500 + 2000(1.1)^{-1} + 2000(1.1)^{-2} = 4971.07$. Since this is less than 5000, the second option is chosen.

**Problem 6–11.** The sinking fund payment must be $320000/s_{\overline{20}|0.04} = 10746.16$, so the annual interest payment is $30000 - 10746.16 = 19253.84$, a yield of $19253.84/320000 = 0.060$.

**Problem 6–12.** The present value of the cost of the first option is $-900/1.09 = -825.69$. The present value of the second option is $1000 - 800 - 1100/1.09 = -809.17$. So the second option is preferred.

**Problem 6–13.** The new payment $N$ must satisfy $Ra_{\overline{30}|0.10} = Ra_{\overline{15}|0.09} + N(1.09)^{15}a_{\overline{5}|0.09}$.

## Solutions to Exercises

**Exercise 6–1.** Exact computation gives the semiannual yield rate as 0.0371 for an annual rate of 0.0742, while the bond salesman's method gives $(6000 + 1000)/(9500 \times 10) = 0.07368$.

**Exercise 6–2.** Here the worst scenario for the investor is that the bond is not called, giving a price of $1000 + (30 - 35)a_{\overline{30}|0.035} = 732.85$.

## §7. Sample Question Set 3

**Solve the following seven problems in no more than 35 minutes.**

**Question 7–1 .** A 1,000 bond with coupon rate $c$ convertible semi-annually will be redeemed at par in $n$ years. The purchase price to yield 5% convertible semi-annually is $P$. If the coupon rate were $c - 0.02$ the price of the bond would be $P - 300$. Another 1,000 bond is redeemable at par at the end of $2n$ years. It has a coupon rate of 7% convertible semi-annually and the yield rate is 5% convertible semi-annually. Calcluate the price of this second bond.

**A.** 1,300

**B.** 1,375

**C.** 1,475

**D.** 2,100

**E.** 2,675

**Question 7–2 .** A 1,000 par value 10 year bond with coupons at 5% convertible semi-annually is selling for 1,081.78. The bond saleman's method is used to approximate the yield rate convertible semi-annually. Calculate the difference between this approximation and the exact value.

**A.** 0.00003

**B.** 0.00006

**C.** 0.00012

**D.** 0.00018

**E.** 0.00024

**Question 7–3 .** A 100 par value 6% bond with semi-annual coupons is purchased at 110 to yield a nominal rate of 4% convertible semi-annually. A similar 3% bond with semi-annual coupons is purchased at $P$ to provide the buyer with the same yield. Calculate $P$.

**A.** 90

**B.** 95

**C.** 100

**D.** 105

**E.** 110

**Question 7–4** . A loan of 1,000 is being repaid in ten years by semi-annual installments of 50, plus interest on the unpaid balance at 4% per annum compounded semi-annually. The installments and interest payments are reinvested at 5% per annum compounded semi-annually. Calculate the annual effective yield rate of the loan.

**A.** 0.046

**B.** 0.048

**C.** 0.050

**D.** 0.052

**E.** 0.054

**Question 7–5** . John buys a 10 year 1,000 par value bond with 8% semi-annual coupons. The price of the bond to earn a yield of 6% convertible semi-annually is 1,204.15. The redemption value is more than the par value. Calculate the price John would have to pay for the same bond to yield 10% convertible semi-annually.

**A.** 875

**B.** 913

**C.** 951

**D.** 989

**E.** 1,027

**Question 7–6** . A 1,000 par value 3 year bond with annual coupons of 50 for the first year, 70 for the second year, and 90 for the third year is bought to yield a force of interest $\delta_t = \dfrac{2t-1}{2(t^2-t+1)}$ for $t \geq 0$. Calculate the price of this bond.

**A.** 500

**B.** 550

**C.** 600

**D.** 650

**E.** 700

**Question 7–7** . The proceeds of a 10,000 death benefit are left on deposit with an insurance company for seven years at an annual effective interest rate of 5%. The balance at the end of seven years is paid to the beneficiary in 120 equal monthly payments of $X$, with the first payment made immediately. During the payout period, interest is credited at an annual effective interest rate of 3%. Calculate $X$.

**A.** 117

**B.** 118

**C.** 129

**D.** 135

**E.** 158

## Answers to Sample Questions

**Question 7–1** . The first two pieces of information give $1000v^{2n} + 1000(c/2)a_{\overline{2n|}} = P$ and $1000v^{2n} + 1000((c - .02)/2)a_{\overline{2n|}} = P - 300$. Using the first in the second gives $P - 10a_{\overline{2n|}} = P - 300$, from which $a_{\overline{2n|}} = 30$. Since the interest rate is 5% convertible semi-annually, $a_{\overline{2n|}} = 40(1 - v^{2n})$, so $v^{2n} = 1/4$. The price of the other bond is $1000v^{4n} + 35a_{\overline{4n|}} = 1000/16 + 35(40)(1 - 1/16) = 1375$. **B**.

**Question 7–2** . The exact yield rate satisfies $25a_{\overline{20|}} + 1000v^{20} = 1081.78$. This gives the exact yield rate as 2% per semi-annual period, or 4% compounded semiannually. The bond salesman's method gives the yield rate as $(20(25) - 81.78))/20817.8 = 0.0200895$ semi-annually. The difference is .0001790. **D**.

**Question 7–3** . The given information means that $3a_{\overline{2n|}} + 100v^{2n} = 110$ and $1.5a_{\overline{2n|}} + 100v^{2n} = P$. Here $a_{\overline{2n|}} = 50(1 - v^{2n})$. Using this the first equation gives $v^{2n} = 4/5$. Using this in the second equation gives $P = 1.5(50)(1 - 4/5) + 100(4/5) = 95$. **B**.

**Question 7–4** . The accumulated amount at the end of 10 years is $50s_{\overline{20|}.025} + (Is)_{\overline{20|}.025} = 1524.56$. The yield rate satisfies $(1 + i/2)^{-20} = 1000$, from which $i = .0426$. **A**.

**Question 7–5** . The redemption value $r$ satisfies $1204.15 = 40a_{\overline{20|}.03} + (1.03)^{-20}r$, from which $r = 1100.01$. The price to yield 10% is $40a_{\overline{20|}.05} + (1.05)^{(}-20)r = 913.07$. **B**.

**Question 7–6** . The price is $50e^{-\int_0^1 \delta_t\,dt} + 70e^{-\int_0^2 \delta_t\,dt} + 1090e^{-\int_0^3 \delta_t\,dt} = 502.40$. **A**.

**Question 7–7** . Since $(.03)^{(12)}/12 = .002466$, $X$ solves $10000(1.05)^7 = X\ddot{a}_{\overline{120|}.002466}$ from which $X = 135.27$. **D**.

## §8. Portfolio Matters, Yield Curves, and Inflation

The yield rate (or dollar weighted rate of interest), as computed earlier, is an important measure for an individual investor to compute when comparing investment alternatives. When used to measure the performance of an investment manager, the yield rate can be misleading. This is because cash inflows and outflows from the fund may not be under the control of the manager. A more meaningful measure of the performance of the investment manager in such situations can be obtained by using **time weighted rates of interest**. The methodology is this. For simplicity, suppose a one year period is to be analyzed and that contributions $C_k$ are made to the fund at times $t_k$, where $0 < t_1 < t_2 < \ldots < t_m = 1$. As before, a negative contribution represents a withdrawal. Denote by $B_0$ the amount in the fund at the beginning of the year and by $B_k$ the amount in the fund immediately *before* the contribution at time $t_k$. The time weighted yield rate $j_k$ over the interval $t_{k-1}$ to $t_k$ is then $1+j_k = B_k/(B_{k-1}+C_{k-1})$. Notice that this yield rate is computed based on the total amount available for investment at time $t_{k-1}$ and is not influenced by the contribution made at time $t_k$. The yield rate $i$ for the entire period is then $1+i = (1+j_1)\ldots(1+j_m)$.

**Example 8–1.** An investment manager's portfolio begins the year with a value of 100,000. Eleven months through the year a withdrawal of 50,000 is made and the value of the portfolio after the withdrawal is 57,000. At the end of the year the value of the portfolio is 60,000. The equation for the time weighted yield rate is $1 + i = (107000/100000)(60000/57000)$ from which $i = 0.1263$. The dollar weighted yield rate satisfies $100000(1 + i) - 50000(1 + i)^{1/12} = 60000$. Linearizing gives the equation $100000(1+i) - 50000(1+i/12) = 60000$, from which $i = 0.1043$.

The other part of the problem caused by inflows and outflows for a managed portfolio is the assignment of returns to individual investors. As an example, suppose investors made contributions to a fund in 2001 when the manager of the fund was able to buy 5 year bonds yielding 7%. Investors depositing funds in 2004 when bonds yielding only 4% can be purchased should not expect a return exceeding 4%, while the original investors should reasonably expect to get about a 7% return, since their money was already invested in the fund. Two approaches to this allocation problem are typically used. Under the **portfolio method** an average rate based on the earnings of the entire fund is computed and credited to each account. In the foregoing discussion, this would cause the early investors to give up part of the income attributed to them in order to subsidize the later investors. The **investment year method** uses an indexing system to connect rates of return to time of investment. After a certain number of years have elapsed from the time of investment, the portfolio method is used on those pooled funds.

Two indexing systems are typically used. The **declining balance system** makes the amount of funds associated with an investment year decline as the monies associated with that investment year are reinvested. The **fixed index system** keeps the funds associated with a given investment year fixed in amount.

Notationally, let $i_t^y$ be the interest rate earned during investment year $t$ on a deposit made in year $y$. Notice that the portfolio method corresponds to the case in which $i_t^y$ does not depend on $t$. Usually there is a number $m$ so that if $t > m$ the portfolio method is used. Notice too that when the portfolio method is used, funds are effectively assumed to be reinvested each year, so that $y$ increases year-by-year.

**Example 8–2.** An investment company was formed in 2000. Suppose the investment year method is applicable for the first 2 years, after which a portfolio rate is used. At the end of 2004, the following rates apply.

| Calendar Year of Investment | Investment Year Rates $i_1$ | $i_2$ | Calendar Year of Portfolio Rate | Portfolio Rate |
|---|---|---|---|---|
| 2000 | 10% | 10% | 2002 | 8% |
| 2001 | 12% | 5% | 2003 | 9% |
| 2002 | 8% | 12% | 2004 | 6% |
| 2003 | 9% | 11% | 2005 | |
| 2004 | 7% | | 2006 | |

An investment of 10000 made in 2001 earned 12% the first year (in 2001), 5% the second year (2002), 9% the third year(2003), and 6% the fourth year (2004). Notice that the portfolio rate was used in 2003 and 2004. An new investment made in 2004 will earn 7% the first year, and so on.

The pattern of returns under the investment year method is not unlike the pattern exhibited by interest rates offered by bonds and other fixed income investments. Typically, the interest rate on short term bonds is lower than the interest rate on longer term bonds. This phenomenon is called the **term structure of interest rates**. The **yield curve** is nothing more than a plot of the rate of return on an investment against the term of the investment. Usually the yield curve has positive slope; if the yield curve has negative slope, the curve is **inverted**. The interest rates appearing in the yield curve are called **spot rates**, since these rates indicate the interest rate to be used today for a transaction terminating at a particular time in the future. Some have argued that the correct way to compute the present value of a future cash stream is to discount future cash flows using the spot rate corresponding to the time at which the cash flow occurs. Of course, the yield curve changes with time. The **forward rate** is the spot rate that will apply at a future time. Forward rates should be considered when an investment made today must be reinvested at some future time, since reinvestment will be made at the appropriate forward rate.

**Example 8–3.** Suppose a payment of 10000 is to be made two years from now.

What is the present value of this payment today if the yield curve shows that the yield on a 2 year bond is 7%? The present value is then $10000(1.07)^{-2}$. Notice that this assumes that the present value could be invested for the full two years in a bond yielding 7%. If for some reason only one year bonds can be used, the appropriate present value would be computed using the one year spot rate and the one year forward rate. If these rates are 6% and 5% respectively, the present value of the 10000 would be $10000(1.06)^{-1}(1.05)^{-1}$. Considering the effect of the term structure of interest rates is a more elaborate analysis than the simple consideration of the problem of reinvestment rates examined earlier.

**Example 8–4.** Suppose that the one year spot rate is 5% and the two year spot rate is 6%. This implies that the one year deferred one year forward rate $f$ satisfies $(1.05)(1+f) = (1.06)^2$, from which $f = 0.070$. Of course, by the time one year passes the yield curve may have shifted and the one year spot rate may have changed.

Associated with any investment involving interest is interest rate risk. This is the risk that interest rates available on new investments will be significantly different from the rate on the investments already made, perhaps in an adverse way. Clearly the interest rate risk increases as the term of the investment increases. But most investments provide cash outflows over the entirety of their life. So what exactly is the term of the investment?

**Example 8–5.** Consider two bonds purchased at the redemption value of 1000, and due in 5 years. One bond pays 5% semi-annually and the other 10% semi-annually. Since the 10% bond returns more cash sooner, the interest rate risk for the 10% bond is less than the interest rate risk for the 5% bond. Can this property be characterized quantitatively?

Consider a general framework in which an investment pays $A_1, \ldots, A_n$ at times $t_1, \ldots, t_n$.

One approach is to compute the average time at which dollars are generated by the investment, and use this average as a more effective characterization of the term of the investment. This approach is the **method of equated time** and gives the investment term as $\sum t_j A_j / \sum A_j$.

**Example 8–6.** In the previous example, the term of the 5% bond is $(25(1/2) + 25(1) + 25(3/2) + \ldots + 25(10) + 1000(10))/1500 = 8.42$ years. The term of the 10% bond is 7.63 years.

A second, and more reasonable, approach is to compute the average time at which dollars *in todays value* are generated by the investment. This measure is called the **duration** (or **Macaulay duration**) and is given by $\sum t_j A_j v^{t_j} / \sum A_j v^{t_j}$. The value of the duration depends on the interest rate used. In the case of bonds, the interest rate used in this computation is usually the yield rate.

**Example 8–7.** Consider again the two bonds and suppose both were purchased to yield 7% compounded semi-annually. The duration of the 5% bond is then 7.80 and the duration of the 10% bond is 6.88.

An alternate viewpoint is also illuminating. The present value of the future payments $P(i) = \sum A_j v^{t_j}$ depends on the interest rate $i$. A reasonable measure of the sensitivity of the investment to interest rate changes is the **volatility** $-P'(i)/P(i)$, which is the percentage change in present value per unit change in interest rate. Direct computation shows that the volatility is the duration divided by $1 + i$. So duration is also measuring the sensitivity of the investment to changes in interest rate. The volatility is also called the **modified duration**.

The duration of a portfolio can be easily determined in terms of the durations of the component investments. Suppose $P_j(i)$ is the present value of the $j$th component of a portfolio. The duration of the portfolio is then

$$(1+i)\frac{\sum_j P_j'(i)}{\sum_j P_j(i)} = \sum_j (1+i)\frac{P_j'(i)}{P_j(i)} \times \frac{P_j(i)}{\sum_j P_j(i)}$$

which is the weighted sum of the individual durations.

Consider a business enterprise which will experience both cash inflows and cash outflows in the conduct of business. Certainly the firm should invest cash on hand, but how should these investments be structured in order to minimize the effect of changes in interest rates when reinvestment of funds must occur? Suppose $A_t$ is the asset cash flow expected at time $t$ and $L_t$ is the liability cash flow expected at time $t$. The surplus for the business when the interest rate is $i$ is then $P(i) = \sum_t (A_t - L_t)(1+i)^{-t}$. The portfolio would be **immunized** against small interest rate changes if $P'(i) = 0$. This means that $\sum_t t(A_t - L_t)/(1+i)^t = 0$, that is, the modified duration of the net cash flow is zero. This fact was first noticed and studied by Frank M. Redington. If in addition $P''(i) > 0$, then small interest rate changes would produce a free profit for a portfolio satisfying $P'(i) = 0$. Of course, this does not occur in practice. The **convexity** is defined by $P''(i)/P(i)$. The portfolio is **fully immunized** if *any* change in interest rate results in a profit. This means that the yield rate produces an absolute minimum of the present value of the net cash flow.

**Example 8–8.** A client deposits 100000 in a bank, with the bank agreeing to pay 8% effective for two years. The client indicates that half of the account balance will be withdrawn at the end of the first year. The bank can invest in either one year or two year zero coupon bonds. The one year bonds yield 9% and the two year bonds yield 10%. Here the cash outflows are 50000(1.08) at the end of the first year and $50000(1.08)^2$ at the end of the second year. Suppose the bank invests $A$ in the one year bonds, and $B$ in two year bonds. Then $P(i) = A(1.09)(1+i)^{-1} - 50000(1.08)(1+i)^{-1} + B(1.10)^2(1+i)^{-2} - 50000(1.08)^2(1+i)^{-2}$. To immunize against interest rate risk at $i = 0.08$, the bank should have $P(0.08) = 0$ and $P'(0.08) = 0$. Solving gives $A = 49641.28$ and $B = 48198.35$. Notice that this allocation gives the bank $1000000 - 49641.28 - 48198.35 = 2260.37$ of profit on the day of deposit. Notice too that this strategy is one of **absolute asset matching**, since the amount invested in one year bonds matures to be exactly equal to the withdrawal at the end of the first year, and the same is true for the investment in two year bonds.

Inflation also affects investments by decreasing the purchasing power of dollars that are received in the future compared to the purchasing power of dollars today. The rate of interest after eliminating the effect of inflation is called the **real** rate of interest, while the actual rate of interest charged in the market is called the **nominal** rate (not to be confused with the different meaning of nominal used earlier). If the real rate of interest is $i'$, the nominal rate is $i$, and the inflation rate is $r$, then $(1+i) = (1+i')(1+r)$. Thus $i = i' + r + i'r$ and $i' = (i-r)/(1+r)$. Most economists believe that $i'$ is relatively stable over time, while $i$ and $r$ increase and decrease more or less in tandem. Notice that if future payments are indexed to inflation, the present value of the future payments using the nominal rate of interest is the same as the present value of *unindexed* payments computed under the real rate of interest $i'$.

## Problems

**Problem 8–1.** Deposits of 10000 are made into an investment fund at time zero and 1. The fund balance at time 1 is 13000 and the balance at time 2 is 21000. Compute the effective yield using both the dollar weighted and the time weighted methods.

**Problem 8–2.** A fund contains 10 on January 1 and receives a deposit of 7 on July 1. The value of the fund immediately before the deposit was 12. The value of the fund on December 31 is 21. Compute the dollar weighted and time weighted rate of return for the year.

**Problem 8–3.** For a certain portfolio, $i_t^y = (1.06 + 0.004t)^{1+0.01y}$ for $1 \leq t \leq 5$ and $0 \leq y \leq 10$. What is the accumulated value at the end of year 4 of a single investment of 1000 at the beginning of year 2? What is the accumulated value at the end of year 4 of a deposit of 1000 at the beginning of each of years 2 through 4?

**Problem 8–4.** An investment fund is started with a deposit of 1000 at time zero, with new deposits being made continuously at a rate of $100 + 20t$ for the next 5 years. The force of interest is $\delta(t) = (1+t)^{-1}$. What is the accumulated amount at the end of 5 years?

**Problem 8–5.** Money is invested for 3 years at an interest rate of 4% effective. If inflation is 5% per year over this period, what percentage of purchasing power is lost?

**Problem 8–6.** Suppose the one year spot rate is 5% and the two year spot rate is 6%. What is the price of a 1000 par two year bond with 5% annual coupons, using these spot rates?

**Problem 8–7.** A 5 year bond with 6% annual coupons has a yield rate of 10% effective and a 5 year bond with 8% annual coupons has a yield rate of 9% effective. What is the 5 year spot rate?

**Problem 8–8.** What is the duration of a stock whose year-end dividends increase by 2% each year if the effective interest rate is 7%?

**Problem 8–9.** A loan with interest rate 5% effective will be repaid with payments of 10000 at the end of the first year, 15000 at the end of the second year and 5000 at the end of the third year. What is the amount of the loan? What is the duration of the loan?

**Problem 8–10.** A loan of 10000 is repaid with equal payments at the end of each year for 10 years. The interest rate is 6%, which is also the yield rate. What is the duration of the loan?

**Problem 8–11.** Find the convexity of a money market fund assuming that the interest rate is 7% effective.

**Problem 8–12.** Find the duration and convexity of a 20 year zero coupon bond assuming that the interest rate is 7% effective.

**Problem 8–13.** A bank agrees to pay 5% compounded annually on a deposit of 100,000 made with the bank. The depositor agrees to leave the funds on deposit on these terms for 8 years. The bank can either buy 4 year zero coupon bonds or preferred stock, both yielding 5% effective. How should the bank apportion its investment in order to immunize itself against interest rate risk?

## Solutions to Problems

**Problem 8–1.** The yield rate $i$ for the dollar weighted method satisfies $10000(1+i)^2 + 10000(1+i) = 21000$, giving $i = 0.0329$. For the time weighted method, $(1+i)^2 = (13000/10000)(21000/23000)$ from which $i = 0.0894$.

**Problem 8–2.** The dollar weighted return satisfies $10(1+i) + 7(1+i)^{1/2} = 21$ giving $i = 0.30$. The time weighted return satisfies $1 + i = (12/10)(21/19) = 1.326$, from which $i = .326$.

**Problem 8–3.** The accumulated value of the single investment is $1000(1.06 + .004)^{1.02}(1.06 + 0.008)^{1.02}(1.06 + .012)^{1.02} = 1222.98$. The second case is the preceding value plus the term $1000(1.064)^{1.03}(1.072)^{1.03} + 1000(1.064)^{1.04} = 2207.36$, for a total of 3430.34.

**Problem 8–4.** A deposit of 1 between time $t$ and $t+dt$ accrues to $e^{\int_t^5 \delta(s)\,ds}$ at time 5, so the total accumulated value is $1000e^{\int_0^5 \delta(s)\,ds} + \int_0^5 (100 + 20t)e^{\int_t^5 \delta(s)\,ds}\,dt = 7460.04$.

**Problem 8–5.** One dollar will accumulate to $(1.04)^3 = 1.1248$ over the period, but the purchasing power will only be $(1.04)^3/(1.05)^3 = 0.9716$, so about 2.84% of the purchasing power of the money is lost.

**Problem 8–6.** The price is $50(1.05)^{-1} + 1050(1.06)^{-2} = 982.11$.

**Problem 8–7.** For convenience, assume both bonds have a par of 1000. Then the prices of the two bonds are $p_1 = 60a_{\overline{5}|.10} + 1000(1.10)^{-5}$ and $p_2 = 80a_{\overline{5}|.09} + 1000(1.09)^{-5}$. These prices can also be computed using spot rates. Thus $(80/60)p_1 - p_2 = (80/60)1060(1+i)^{-5} - 1080(1+i)^{-5}$ where $i$ is the five year spot rate. Solving gives the five year spot rate as 0.1440.

**Problem 8–8.** The duration is $\sum_{k=1}^{\infty} k(1.02)^{k-1}(1.07)^{-k} / \sum_{k=1}^{\infty}(1.02)^{k-1}(1.07)^{-k} = 21.40$.

**Problem 8–9.** The amount of the loan is $10000(1.05)^{-1} + 15000(1.05)^{-2} + 5000(1.05)^{-3} = 27448.44$. The duration is

$$\frac{10000(1.05)^{-1} + 30000(1.05)^{-2} + 15000(1.05)^{-3}}{27448.44} = 1.81.$$

**Problem 8–10.** The duration is $(Ia)_{\overline{10}|0.06}/a_{\overline{10}|0.06} = 5.02$. Notice that the amount of the loan and the interest rate on the loan are both irrelevant. Only the yield rate is used.

**Problem 8–11.** The money market fund throws off cash at a constant rate per unit time. Each dollar invested will return $i$ at the end of each period, forever. Thus $S(i) = \sum_{k=1}^{\infty} i(1+i)^{-k} = 1$. Thus the convexity is zero.

**Problem 8–12.** The duration is 20 since the par value is repaid at the end of 20 years and this is the only payment made by the bond. Now, per dollar of par value, $S(i) = (1+i)^{-20}$, so that the convexity $S''(i)/S(i) = 420/(1+i)^2$. Substituting $i = 0.07$ gives the convexity as 366.844.

**Problem 8–13.** The present value of the cash flows in terms of the amount $B$ invested in the bonds is $P(i) = B(1.05)^4(1+i)^{-4} + (100000 - B)(0.05)/i - 100000(1.05)^8(1+i)^{-8}$ since the preferred stock is assumed to pay 5% forever and the bank must repay the deposit with interest at the end of 8 years. Substitution verifies that $P(.05) = 0$. Setting $P'(.05) = 0$ gives $B = 76470.58$ as the amount to be invested in bonds with the remainder in stock. With this allocation $P''(.05) > 0$, so the portfolio is optimal.

## §9. Sample Question Set 4

**Solve the following 7 problems in no more than 35 minutes.**

**Question 9–1 .** An executive is given a buyout package by the company that will pay a monthly benefit for the next 20 years. The benefit will remain constant within each of the 20 years. At the end of each 12 month period the monthly benefits will be adjusted upwards to reflect the percentage increase in the CPI. The first monthly payment is $R$ and will be paid one month from today. The CPI increases a 3.2% per year forever. At an annual effective interest rate of 6% the buyout package has a value of 100,000. Calculute $R$.

**A.** 517

**B.** 538 **D.** 548

**C.** 540 **E.** 563

**Question 9–2 .** On January 1, 1997 an investment account is worth 100,000. On April 1, 1997 the value has increased to 103,000 and 8,000 is withdrawn. On January 1, 1999 the account is worth 103,992. Assuming a dollar weighted method for 1997 and a time weighted method for 1998 the annual effective interest rate was equal to $x$ for both 1997 and 1998. Calculate $x$.

**A.** 6.00%

**B.** 6.25% **D.** 6.75%

**C.** 6.50% **E.** 7.00%

**Question 9–3 .** Betty borrows 19,800 from bank X. Betty repays the loan by making 36 equal payments of principal each month. She also pays interest on the unpaid balance each month at a nominal rate of 12% compounded monthly. Immediately after the 16th payment is made, Bank X sells the rights to future payments to Bank Y. Bank Y wishes to yield a nominal rate of 14% compounded semi-annually on its investment. What price did Bank X receive?

**A.** 9,792

**B.** 10,823 **D.** 11,671

**C.** 10,857 **E.** 11,709

**Question 9–4** . A firm has proposed the following restructuring for one of its 1000 par value bonds. The bond presently has 10 years remaining until maturity. The coupon rate on the existing bond is 6.75% per annum paid semiannually. The current nominal semiannual yield on the bond is 7.40%. The company proposes suspending coupon payments for four years with the suspended coupon payments being repaid, with accrued interest, when the bond comes due. Accrued interest is calculated using a nominal semiannual rate of 7.40%. Calculate the market value of the restructured bond.

**A.** 755

**B.** 805

**C.** 855

**D.** 905

**E.** 955

**Question 9–5** . The price of a share of stock issued by a company is 30 on January 1, 2000. The price assumes annual dividends, with the first dividend due on December 31, 2000. The first dividend will be 1.65 with future dividends expected to grow at a rate of 3% per year forever. Which of the following statements are true, assuming the market capitalization rate remains the same in future years?
I. If an investor requires an annual effective return of at least 9% on all its asset purchases, it will consider purchasing the common stock of this company on January 1, 2000.
II. If an investor buys 100 shares of stock on January 1, 2000 and later sells all 100 shares on January 1, 2003, the investor will expect a capital gain of 278.18.
III. On July 1, 2000 the price of the stock will be 1.5% greater than the price on January 1, 2000.

**A.** II only

**B.** III only

**C.** I and II only

**D.** I and III only

**E.** II and III only

**Question 9–6** . A customer is offered an investment where interest is calculated according to the force of interest $\delta_t = 0.02t$ for $0 \le t \le 3$ and $\delta_t = 0.045$ for $t > 3$. The customer invests 1000 at time $t = 0$. What nominal rate of interest, compounded quarterly, is earned over the first four year period?

**A.** 3.4%

**B.** 3.7%

**C.** 4.0%

**D.** 4.2%

**E.** 4.5%

**Question 9–7** . Seth deposits $X$ in an account today in order to fund his retirement. He would like to receive payments of 50 per year, in real terms, at the end of each year for a total of 12 years, with the first payment occurring seven years from now. The inflation rate will be 0.0% for the next six years and 1.2% per annum thereafter. The annual effective rate of return is 6.3%. Calculate $X$.

**A.** 303

**B.** 306

**C.** 316

**D.** 327

**E.** 329

## Answers to Sample Questions

**Question 9–1** . Solving $100000 = Ra_{\overline{12}|.06^{(12)}/12}(1 + \sum_{k=1}^{19}(1.032)^k/(1.06)^k)$ gives $R = 547.92$. **D**.

**Question 9–2** . Let $A$ be the amount in the account on January 1 1998. The dollar weighted scheme gives $100000(1 + x) - 8000(1 + (8/12)x) = A$, while the time weighted scheme gives $A(1 + x) = 103992$. Solving the second equation for $A$ and substituting into the first gives $x = 0.0622$. **B**.

**Question 9–3** . Betty's monthly payment is $19800/36 = 550$ plus 1% of the loan balance. The amount of the loan after the 16th payment is $19800-550(16) = 11,000$. Betty's $k$th payment going forward will be $550+110-5.50(k-1) = 665.50-5.50k$, $1 \le k \le 20$. The present value of this stream at an interest rate of 14% compounded semiannually is $665.50a_{\overline{20}|0.0113} - 5.50(Ia)_{\overline{20}|0.0113} = 10861.52$, which is the amount Bank X receives. **C**.

**Question 9–4** . The accrued coupons have value $33.75s_{\overline{8}|.037}(1.037)^{12} = 475.81$. So the value of the bond is $1475.81(1.037)^{-20} + 33.75a_{\overline{12}|.037}(1.037)^{-8} = 954.63$. **E**.

**Question 9–5** . From the given information, $30 = \sum_{k=1}^{\infty} 1.65(1.03)^{k-1}(1 + i)^{-k}$ from which the valuation interest rate is $i = 0.085$. Thus I is not true, since a valuation rate of 9% would produce a lower stock price. The price on January 1, 2003 would be 32.781, so II is true. The price on July 1, 2000 would be 31.248, so III is false. **A**.

**Question 9–6** . After 4 years the account holds $1000e^{\int_0^3 \delta_t\,dt}e^{\int_3^4 .045\,dt} = 1144.53$. Solving $1000(1 + i/4)^{16} = 1144.53$ for $i$ gives $i = 0.03389$. **A**.

**Question 9–7** . Clearly $X = (1.063)^{-7}(1.012)\sum_{k=0}^{11} 50(1.012/1.063)^k = 306.47$. **B**.

## §10. Practice Examinations

The remaining sections contain practice examinations. These were constructed from old Course 140 examinations of the Society of Actuaries and also some of the relevant parts of Course 2 examinations. Each practice examination consists of 8 questions and should be completed in no more than 40 minutes.

## §11. Practice Examination 1

**1**. A bank customer borrows $X$ at an annual effective rate of 12.5% and makes level payments at the end of each year for $n$ years. The interest portion of the final payment is 153.86. The total principal repaid as of time $(n-1)$ is 6009.12. The principal repaid in the first payment is $Y$. Calculate $Y$.

**A.** 470

**B.** 480

**C.** 490

**D.** 500

**E.** 510

**2**. A perpetuity paying 1 at the beginning of each 6-month period has a present value of 20. A second perpetuity pays $X$ at the beginning of every 2 years. Assuming the same annual effective interest rate, the two present values are equal. Determine $X$.

**A.** 3.5

**B.** 3.6

**C.** 3.7

**D.** 3.8

**E.** 3.9

**3**. A business takes out a loan of 12,000 at a nominal rate of 12% compounded quarterly. Payments of 750 are made at the end of every 6 months for as long as necessary to pay back the loan. Three months before the ninth payment is due, the company refinances the loan at a nominal rate of 9% compounded monthly. Under the refinanced loan payments of $R$ are to be made monthly with the first payment to be made at the same time that the ninth payment under the old loan was to be made. A total of 30 monthly payments will completely pay off the loan. Determine $R$.

**A.** 448

**B.** 452

**C.** 456

**D.** 461

**E.** 465

**4**. Sally lends 10,000 to Tim. Tim agrees to pay back the loan over 5 years with monthly payments payable at the end of each month. Sally can reinvest the monthly payments from Tim in a savings account paying interest at 6% compounded monthly. The yield rate earned on Sally's investment over the five year period turned out to be 7.45% compounded semi-annually. What nominal rate of interest, compounded monthly, did Sally charge Tim on the loan?

**A.** 8.53%

**B.** 8.59%

**C.** 8.68%

**D.** 8.80%

**E.** 9.16%

**5**. A 1000 par value 5 year bond with 8% semiannual coupons was bought to yield 7.5% convertible semiannually. Determine the amount of premium amortized in the sixth coupon payment.

**A.** 2.00

**B.** 2.08

**C.** 2.15

**D.** 2.25

**E.** 2.34

**6**. Jim began saving money for his retirement by making monthly deposits of 200 into a fund earning 6% interest compounded monthly. The first deposit occurred on January 1, 1985. Jim became unemployed and missed making deposits 60 through 72. He then continued making monthly deposits of 200. How much did Jim accumulate in his fund on December 31, 1999?

**A.** 53,572

**B.** 53,715

**C.** 53,840

**D.** 53,966

**E.** 54,184

**7**. Investment $X$ for 100,000 is invested at a nominal rate of interest $j$ convertible semi-annually. After four years it accumulates to 214,358.88. Investment $Y$ for 100,000 is invested at a nominal rate of discount $k$ convertible quarterly. After two years, it accumulates to 232,305.73. Investment $Z$ for 100,000 is invested at an annual effective rate of interest equal to $j$ in year one and an annual effective rate of discount equal to $k$ in year two. Calculate the value of investment $Z$ at the end of two years.

**A.** 168,000

**B.** 182,900

**C.** 184,425

**D.** 200,000

**E.** 201,675

**8**. Which of the following are true?

I. $\frac{d}{dd}(i) = v^{-2}$

II. $\frac{d}{di}(i^{(m)}) = v^{-\left(\frac{m-1}{m}\right)}$

III. $\frac{d}{d\delta}(i) = 1 + i$

**A.** I and II only

**B.** I and III only

**C.** II and III only

**D.** I, II, and III

**E.** The correct answer is not given by A, B, C, or D

## Solutions to Practice Examination 1

**1**. The loan balance after $(n-1)$ payments is $Xv/a_{\overline{n}|}$, so the information about the interest portion of the final payment gives $0.125Xv/a_{\overline{n}|} = 153.86$. Thus the annual payment is $X/a_{\overline{n}|} = 1384.74$. The original amount of the loan is therefore $X = 6009.12 + 1384.74 - 153.86 = 7240$. The interest part of the first payment is $7240(0.125) = 905.00$, leaving the principal part as $1384.74 - 905.00 = 479.74$. **B**.

**2**. The equations of value are $\sum_{k=0}^{\infty}(1+i)^{-k/2} = 20$ and $20 = X\sum_{k=0}^{\infty}(1+i)^{-2k}$, from which $\dfrac{1}{\sqrt{1+i}} = 19/20$, $(1+i)^{-2} = (19/20)^4 = .8145$, and $X = 3.708$. **C**.

**3**. The loan balance on the old loan just after the eighth payment is $12000(1.03)^{16} - 750s_{\overline{8}|.0609} = 11809.35$. This balance is carried forward 3 months at the old rate to get the amount refinanced as 12163.63. Carrying this forward another 2 months at 9% compounded monthly will get the balance to be repaid by installments, which is 12346.77, giving the monthly installment as 461.13. **D**.

**4**. The equation of value is $10000(1+.0745/2)^{10} = Ps_{\overline{60}|.005}$ from which $P = 206.61$. Thus $a_{\overline{60}|i/12} = 48.40$, and $i = 0.0879$. **D**.

**5**. The price of the bond is $1000 + 2.50a_{\overline{10}|.0375} = 1020.53$. Thus each amortization is $20.53/a_{\overline{10}|.0375} = 2.50$, and the balance after the fifth coupon is $2.50a_{\overline{5}|.0375} = 11.208$, making the interest part of the sixth coupon $11.208(.0375) = 0.42$ and the principal part 2.08. **B**.

**6**. The accumulated amount is $200\ddot{s}_{\overline{59}|.005}(1.005)^{121} + 200\ddot{s}_{\overline{108}|.005} = 53839.83$. **C**.

**7**. From the information given, $100000(1+j/2)^8 = 214358.88$ from which $j = 0.20$, and $100000(1-k/4)^{-8} = 232305.73$ from which $k = 0.40$, so $100000(1+j)(1-k)^{-1} = 200000$. **D**.

**8**. Since $i = (1-d)^{-1} - 1$, $\frac{d}{dd}i = (1-d)^{-2} = v^{-2}$, so I holds. Also $\frac{d}{di}i^{(m)} = \frac{d}{di}m((1+i)^{1/m} - 1) = (1+i)^{1/m-1} = v^{(m-1)/m}$, so II does not hold. Since $i = e^{\delta} - 1$, $\frac{d}{d\delta}i = e^{\delta} = 1+i$ and III holds. **B**.

# §12. Practice Examination 2

**1**. A fund starts with a zero balance at time zero. The fund accumulates with a varying force of interest $\delta_t = \dfrac{2t}{t^2+1}$ for $t \geq 0$. A deposit of 100,000 is made at time 2. Calculate the number of years from the time of deposit for the fund to double.

**A.** 0.5

**B.** 1.0

**C.** 1.5

**D.** 2.0

**E.** 2.5

**2**. Two funds, $X$ and $Y$, start with the same amount. You are given

(1) Fund $X$ accumulates at a force of interest 5%

(2) Fund $Y$ accumulates at a rate of interest $j$ compounded semi-annually

(3) At the end of eight years, Fund $X$ is 1.05 times as large as Fund $Y$

Calculate $j$.

**A.** 0.022

**B.** 0.023

**C.** 0.042

**D.** 0.044

**E.** 0.046

**3**. Carl puts 10,000 into a bank account that pays an annual effective interest rate of 4% for ten years. If a withdrawal is made during the first five and one-half years, a penalty of 5% of the withdrawal amount is made. Carl withdraws $K$ at the end of each of years 4, 5, 6, and 7. The balance in the account at the end of year 10 is 10,000. Calculate $K$.

**A.** 929

**B.** 958

**C.** 980

**D.** 1,005

**E.** 1,031

**4**. You are given that $X$ is the current value at the end of year two of a 20 year annuity due of 1 per annum and that the effective interest rate for year $t$ is $\frac{1}{8+t}$. Calculate $X$.

**A.** $\sum_{t=9}^{28} \frac{10}{t}$

**B.** $\sum_{t=9}^{28} \frac{11}{t}$

**C.** $\sum_{t=10}^{29} \frac{10}{t}$

**D.** $\sum_{t=10}^{29} \frac{11}{t}$

**E.** $\sum_{t=11}^{30} \frac{10}{t}$

**5**. Gloria borrows 100,000 to be repaid over 30 years. You are given

(1) Her first payment is $X$ at the end of year 1

(2) Her payments increase at the rate of 100 per year for the next 19 years and remain level for the following 10 years

(3) The effective rate of interest is 5% per annum

Calculate $X$.

**A.** 5,505

**B.** 5,555

**C.** 5,605

**D.** 5,655

**E.** 5,705

**6**. You are given a perpetuity, with annual payments as follows:

(1) Payments of 1 at the end of the first year and every three years thereafter

(2) Payments of 2 at the end of the second year and every three years thereafter

(3) Payments of 3 at the end of the third year and every three years thereafter

(4) The interest rate is 5% convertible semi-annually

Calculate the present value of this perpetuity.

**A.** 24

**B.** 29

**C.** 34

**D.** 39

**E.** 47

**7**. You are given a perpetual annuity immediate with annual payments increasing in geometric progression, with a common ratio of 1.07. The annual effective interest rate is 12%. The first payment is 1. Calculate the present value of this annuity.

**A.** 18

**B.** 19

**C.** 20

**D.** 21

**E.** 22

**8**. A loan of 1,000 is taken out at an annual effective interest rate of 5%. Level annual interest payments are made at the end of each year for 10 years, and the principal amount is repaid at the end of 10 years. At the end of each year, the borrower makes level annual payments to a sinking fund that earns interest at an annual effective rate of 4%. At the end of 10 years the sinking fund accumulates to the loan principal. Calculate the difference between the interest payment on the loan and the interest earned by the sinking fund in the fifth year.

**A.** 23

**B.** 26

**C.** 33

**D.** 36

**E.** 38

## Solutions to Practice Examination 2

**1**. The time $t$ at which the funds are doubled is the solution of $e^{\int_2^t \delta_s\,ds} = 2$, which gives $(t^2+1)/5 = 2$ from which $t = 3$. Hence only one year of investment is required. **B**.

**2**. The equation of value is $e^{.05(8)} = 1.05(1 + j/2)^{16}$ from which $j = 0.0443$. **D**.

**3**. The equation of value at the end of the fourth year is $10000v^6 = 10000(1.04)^4 - (1.05)K - 1.05Kv - Kv^2 - Kv^3$, from which $K = 979.93$. **C**.

**4**. The discount factor for year $t$ is $(1 + 1/(8+t))^{-1} = \frac{8+t}{9+t}$. Products of these factors telescope. So the present value of the twenty annuity payments is $11/9 + 11/10 + 11/11 + 11/12 + 11/13 + \ldots + 11/28$. **B**.

**5**. Here $100000 = Xa_{\overline{30|}} + v100(Ia)_{\overline{19|}} + v^{20}1900a_{\overline{10|}}$, from which $X = 5504.76$. **A**.

**6**. First principles gives the present value as $(r^{-2} + 2r^{-4} + 3r^{-6})\sum_{j=0}^{\infty} r^{-6j}$ where $r = (1 + .05/2)$. Simplification gives the value as 38.85. **D**.

**7**. The present value is $\sum_{j=1}^{\infty}(1.07)^{j-1}/(1.12)^j = 20$. **C**.

**8**. The annual sinking fund contribution is $1000/s_{\overline{10|}.04} = 83.29$. The balance in the sinking fund at the beginning of the fifth year is $83.29s_{\overline{4|}.04} = 353.69$ which earns 14.15 in interest during the fifth year. Since the interest on the loan is 50 per year the difference is 35.85. **D**.

## §13. Practice Examination 3

**1**. A loan is to be amortized by $n$ level annual payments of $X$ where $n > 5$. You are given

(1) The amount of interest in the first payment is 604.00

(2) The amount of interest in the third payment is 593.75

(3) The amount of interest in the fifth payment is 582.45

Calculate $X$.

**A.** 704

**B.** 739

**C.** 1,163

**D.** 1,198

**E.** 1,233

**2**. Tom borrows 100 at an annual effective interest rate of 4% and agrees to repay it with 30 annual installments. The amount of each payment in the last 20 years is set at twice that in the first 10 years. At the end of 10 years, Tom has the option to repay the entire loan with a final payment $X$, in addition to the regular payment. This will yield the lender an annual effective rate of 4.5% over the 10 year period. Calculate $X$.

**A.** 89

**B.** 94

**C.** 99

**D.** 104

**E.** 109

**3**. You are given two $n$-year par value 1,000 bonds. Bond $X$ has 14% semi-annual coupons and a price of 1,407.70 to yield $i$ compounded semi-annually. Bond $Y$ has 12% semi-annual coupons and a price of 1,271.80 to yield the same rate $i$ compounded semi-annually. Calculate the price of Bond $X$ to yield $i - 1\%$.

**A.** 1,500

**B.** 1,550

**C.** 1,600

**D.** 1,650

**E.** 1,700

**4**. Henry has a five year 1,000,000 bond with coupons at 6% convertible semi-annually. Fiona buys a 10 year bond with face amount $X$ and coupons at 6% convertible semi-annually. Both bonds are redeemable at par. Henry and Fiona both buy their bonds to yield 4% compounded semi-annually and immediately sell them to an investor to yield 2% compounded semi-annually. Fiona earns the same amount of profit as Henry. Calculate $X$.

**A.** 500,000

**B.** 502,000

**C.** 505,000

**D.** 571,000

**E.** 574,000

**5**. On January 1 of each year Company ABC declares a dividend to be paid quarterly on its common shares. Currently, 2 per share is paid at the end of each calendar quarter. Future dividends are expected to increase at the rate of 5% per year. On January 1 of this year, an investor purchased some shares at $X$ per share, to yield 12% convertible quarterly. Calculate $X$.

**A.** 103

**B.** 105

**C.** 107

**D.** 109

**E.** 111

**6**. Five years ago, the XYZ company bought a 20 year 100,000 non-callable bond with coupons at 8% convertible semi-annually. The next coupon is due six months from today. You are given

(1) The bond was bought to yield 7% compounded semi-annually

(2) The market value is based on a 6% interest rate, compounded semi-annually

(3) The book value is equal to the adjusted cost (amortized value) of the bond

Calculate the unrealized capital gain.

**A.** 9,200

**B.** 10,400

**C.** 18,400

**D.** 19,600

**E.** 20,800

**7**. An investor puts 100 into Fund $X$ and 100 into Fund $Y$. Fund $Y$ earns compound interest at the annual rate of $j > 0$, and Fund $X$ earns simple interest at the annual rate of $1.05j$. At the end of 2 years, the amount in Fund $Y$ is equal to the amount in Fund $X$. Calculate the amount in Fund $Y$ at the end of 5 years.

**A.** 150

**B.** 153

**C.** 157

**D.** 161

**E.** 165

**8**. You are given a loan on which interest is charged over a 4 year period, as follows

(1) an effective rate of discount of 6% for the first year

(2) a nominal rate of discount of 5% compounded every 2 years for the second year

(3) a nominal rate of interest of 5% compounded semiannually for the third year

(4) a force of interest of 5% for the fourth year

Calculate the annual effective rate of interest over the 4 year period.

**A.** 0.0500

**B.** 0.0525

**C.** 0.0550

**D.** 0.0575

**E.** 0.0600

## Solutions to Practice Examination 3

**1**. If $A$ is the loan amount, the amount interest in the first payment is $iA$, in the third payment is $iAa_{\overline{n-2}|}/a_{\overline{n}|}$ and in the fifth payment is $iAa_{\overline{n-4}|}/a_{\overline{n}|}$. Note that $X = A/a_{\overline{n}|}$, so these amounts are $X(1-v^n)$, $X(1-v^{n-2})$ and $X(1-v^{n-4})$. Thus $(X-604)/(X-593.75) = (X-593.75)/(X-582.45)$, from which $X = 704.05$. **A**.

**2**. The equations of value are $100 = Aa_{\overline{10}|.04} + v^{10}2Aa_{\overline{20}|.04}$ and $100 = Aa_{\overline{10}|.045} + v^{10}X$. From the first, $A = 3.77$, and from the second $X = 108.97$. **E**.

**3**. The given information yields $1000+(70-500i)a_{\overline{2n}|i/2} = 1407.70$ and $1000+(60-500i)a_{\overline{2n}|i/2} = 1271.80$. Substracting 1000 from both sides and then taking ratios gives $(70-500i)/(60-500i) = 407.70/271.80$ and $i = 0.08$. Using this value of $i$ in the first equation gives $n = 10$. So the value of Bond $X$ to yield 7% is 1497.43. **A**.

**4**. The profit for Henry is $1000000 + (30000-10000)a_{\overline{10}|.01} - (1000000 + (30000-20000)a_{\overline{10}|.02}) = 99600.02$ while Fiona's profit is $X + (.03X - .01X)a_{\overline{20}|.01} - (X + (.03X - .02X)a_{\overline{20}|.02})$. Equating gives $X = 504,568.85$. **C**.

**5**. Clearly $X = 2a_{\overline{4}|.03}\sum_{j=0}^{\infty}(1.05)^j/(1.03)^{4j} = 110.81$. **E**.

**6**. The book value today is $100000 + 500a_{\overline{30}|.035} = 109196.02$. The market value today is $100000+1000a_{\overline{30}|.03} = 119600.44$. The difference between the market value and the book value is $119600 - 109196 = 10404$. **B**.

**7**. The equation of value is $100(1+j)^2 = 100(1+2(1.05j))$ from which $j = 0.1$. So the amount after 5 years in Fund Y is $100(1+0.1)^5 = 161.05$. **D**.

**8**. The effective rate of interest $i$ satisfies $(1+i)^4 = (1-.06)^{-1}(1-.05/(1/2))^{-1/2}(1+.05/2)^2e^{.05} = 1.2385$, from which $i = 0.0549$. **C**.

## §14. Practice Examination 4

**1**. Jim buys a perpetuity of 100 per year, with the first payment 1 year from now. The price for the perpetuity is 975.61, based on a nominal yield of $i$ compounded semiannually. Immediately after the second payment is received, the perpetuity is sold for 1642.04, to earn for the buyer a nominal yield of $j$ compounded semiannually. Calculate $i - j$.

**A.** 0.020

**B.** 0.030

**C.** 0.035

**D.** 0.040

**E.** 0.042

**2**. You are given

(1) Fund $X$ accumulates at an interest rate of 8% compounded quarterly

(2) Fund $Y$ accumulates at an interest rate of 6% compounded semiannually

(3) at the end of 10 years, the total amount in the two funds combined is 1000

(4) at the end of 5 years, the amount in Fund $X$ is twice that in Fund $Y$

Calculate the total amount in the two funds at the end of 2 years.

**A.** 560

**B.** 570

**C.** 580

**D.** 590

**E.** 600

**3**. You are given $\delta_t = \dfrac{1}{1+t}$ for $0 \le t \le 5$. Calculate $s_{\overline{5}|}$.

**A.** 7.5

**B.** 8.7

**C.** 10.5

**D.** 13.7

**E.** 16.0

**4**. At an effective annual interest rate $i$, you are given

(1) the present value of an annuity immediate with annual payments of 1 for $n$ years is 40

(2) the present value of an annuity immediate with annual payments of 1 for $3n$ years is 70

Calculate the accumulated value of an annuity immediate with annual payments of 1 for $2n$ years.

**A.** 240

**B.** 243

**C.** 260

**D.** 268

**E.** 280

**5**. A 20 year annuity pays 100 every other year beginning at the end of the second year, with additional payments of 300 each at the ends of years 3, 9, and 15. The effective annual interest rate is 4%. Calculate the present value of the annuity.

**A.** 1310

**B.** 1340

**C.** 1370

**D.** 1400

**E.** 1430

**6**. An annuity immediate pays 10 at the ends of years 1 and 2, 9 at the ends of years 3 and 4, etc., with payments decreasing by 1 every second year until nothing is paid. The effective annual rate of interest is 5%. Calculate the present value of this annuity immediate.

**A.** 71

**B.** 78

**C.** 84

**D.** 88

**E.** 94

**7**. On June 1, 1990, an investor buys three 14 year bonds, each with par value 1000, to yield an effective annual interest rate of $i$ on each bond. Each bond is redeemable at par. You are given

(1) the first bond is an accumulation bond priced at 195.63

(2) the second bond has 9.4% semiannual coupons and is priced at 825.72

(3) the third bond has 10% annual coupons and is priced at $P$

Calculate $P$.

**A.** 825

**B.** 835

**C.** 845

**D.** 855

**E.** 865

**8**. A 10 year loan with an effective annual interest rate of 5% is to be repaid with the following payments

(1) 100 at the end of the second year

(2) 200 at the end of the fourth year

(3) 300 at the end of the sixth year

(4) 400 at the end of the eighth year

(5) 500 at the end of the tenth year

Calculate the amount of interest included in the second payment.

**A.** 100

**B.** 103

**C.** 105

**D.** 109

**E.** 114

## Solutions to Practice Examination 4

**1**. The equations of value are $975.61 = \sum_{k=1}^{\infty}(1+i/2)^{-2k}$ and $1642.04 = \sum_{k=1}^{\infty}(1+j/2)^{-2k}$. Thus $i = 0.10$ and $j = 0.06$. **D**.

**2**. Let $X$ and $Y$ denote the amount initially in the two funds. Then $X(1.02)^{40} + Y(1.03)^{20} = 1000$ and $X(1.02)^{20} = 2Y(1.03)^{10}$. Solving the second for $X$ in terms of $Y$ and substituting into the first gives $X = 311.86$ and $Y = 172.41$. Thus $X(1.02)^8 + Y(1.03)^4 = 559.44$. **A** .

**3**. Direct computation gives $s_{\overline{5}|} = 1e^{\int_1^5 \delta_t\,dt} + \ldots + 1e^{\int_4^5 \delta_t\,dt} = (6/2) + (6/3) + (6/4) + (6/5) + (6/6) = 8.7$. **B**.

**4**. First note that $a_{\overline{3n}|} = a_{\overline{n}|}(1 + v^n + v^{2n})$, giving $7/4 = 1 + v^n + v^{2n}$. Solving gives $v^n = 1/2$. Second, $a_{\overline{3n}|} = a_{\overline{n}|} + v^n a_{\overline{2n}|}$ so that $v^n a_{\overline{2n}|} = 30$. Hence $s_{\overline{2n}|} = v^{-2n} a_{\overline{2n}|} = 240$. **A**.

**5**. Direct reasoning gives the present value as $100\sum_{j=1}^{10}(1.04)^{-2j} + 300(v^3 + v^9 + v^{15}) = 1310.24$. **A**.

**6**. The present value is $(v^{-1} + 1)(Da)_{\overline{10}|0.1025} = 78.41$. **B**.

**7**. Since the first bond has no coupons, $1000(1+i)^{-14} = 195.63$. From which $i = 0.1236$. For the third bond, $P = 100a_{\overline{14}|i} + 1000(1+i)^{-14} = 846.41$. **C**.

**8**. The loan balance at the end the second year is $200v^2 + 300v^4 + 400v^6 + 500v^8 = 1065.12$, on which two years of interest is 109.18. **D**.

## §15. Practice Examination 5

**1**. A loan is to be repaid by annual installments of $X$ at the end of each year for 10 years. You are given

(1) the total principal repaid in the first 3 years is 290.35

(2) the total principal repaid in the last 3 years is 408.55

Calculate the total amount of interest paid during the life of the loan.

**A.** 300

**B.** 320

**C.** 340

**D.** 360

**E.** 380

**2**. Bob takes out a loan of 1000 at an annual effective interest rate of $i$. You are given

(1) the first payment is made at the end of year 6

(2) ten equal annual payments are made to repay the loan in full at the end of 15 years

(3) the outstanding principal after the payment made at the end of year 10 is 908.91

Calculate the outstanding principal at the end of year 5.

**A.** 1390

**B.** 1420

**C.** 1450

**D.** 1480

**E.** 1510

**3**. On May 1, 1985, a bond with par value 1000 and annual coupons at 5.375% was purchased to yield an effective annual interest rate of 5%. On May 1, 2000, the bond is redeemable at 1100. The book value of the bond is adjusted each year so that it equals the redemption value on May 1, 2000. Calculate the amount of write-up or write-down in the book value in the year ending May 1, 1991.

**A.** 1.25 write-down

**B.** 0.81 write-down

**C.** 0.77 write-down

**D.** 0.81 write-up

**E.** 0.77 write-up

**4**. You are given

(1) the present value of a $6n$ year annuity immediate of 1 at the end of every year is 9.996

(2) the present value of a $6n$ year annuity immediate of 1 at the end of every second year is 4.760

(3) the present value of a $6n$ year annuity immediate of 1 at the end of every third year is $X$

Calculate $X$.

**A.** 2.87

**B.** 3.02

**C.** 3.17

**D.** 3.32

**E.** 4.17

**5**. A 10 year bond with par value 1000 and annual coupon rate $r$ is redeemable at 1100. You are given

(1) the price to yield an effective annual interest rate of 4% is $P$

(2) the price to yield an effective annual interest rate of 5% is $P - 81.49$

(3) the price to yield an effective annual interest rate of $r$ is $X$

Calculate $X$.

**A.** 1061

**B.** 1064

**C.** 1068

**D.** 1071

**E.** 1075

**6**. Paul borrows 1000 at an annual interest rate of 12%. He repays the loan in full by making the following payments:

(1) 500 at the end of 4 months

(2) 200 at the end of 14 months

(3) $R$ at the end of 18 months

Calculate $R$.

**A.** 402

**B.** 405

**C.** 407

**D.** 410

**E.** 414

**7**. A fund earns interest at a force of interest $\delta_t = kt$. A deposit of 100 at time $t = 0$ will grow to 250 at the end of 5 years. Determine $k$.

**A.** $0.08\ln(1.5)$

**B.** 0.06

**C.** $0.08\ln(2.5)$

**D.** 0.08

**E.** $\ln(2.5)$

**8**. You are given that $a_{\overline{n}|} = 10.00$ and $a_{\overline{3n}|} = 24.40$. Determine $a_{\overline{4n}|}$.

**A.** 28.74

**B.** 29.00

**C.** 29.26

**D.** 29.52

**E.** 29.78

## Solutions to Practice Examination 5

**1**. Now $Xa_{\overline{10}|} - Xa_{\overline{7}|} = 290.35$ and $Xa_{\overline{3}|} = 408.55$. Substituting $a_{\overline{10}|} = a_{\overline{7}|} + v^7 a_{\overline{3}|}$ into the first fact and using the second gives $v^7 = 290.35/408.55$ and $i = 0.05$. Using this in the second fact gives $X = 150.02$ and the total of all of the payments is thus 1500.20, of which $Xa_{\overline{10}|.05} = 1158.43$ is principal, leaving $1500.20 - 1158.43 = 341.77$ as interest. **C**.

**2**. The given information yields $pv^5 a_{\overline{10}|} = 1000$ and $pa_{\overline{5}|} = 908.91$. Substituting $a_{\overline{10}|} = a_{\overline{5}|} + v^5 a_{\overline{5}|}$ into the first equation and using the second gives $v^5 908.91(1+v^5) = 1000$. Solving this equation gives $v^5 = 0.661$ and the balance at the end of year 5 is $pa_{\overline{10}|} = 1000/0.661 = 1512.86$. **E**.

**3**. The purchase price of the bond was $53.75a_{\overline{15}|.05} + 1100(1.05)^{-15} = 1087.03$, which is a discount to the redemption value. The book value on May 1, 1990 was $53.75a_{\overline{10}|.05} + 1100(1.05)^{-10} = 1090.35$, on which 5% interest is 54.52, leaving a write-up of $54.52 - 53.75 = 0.77$. **E**.

**4**. Let the present value of the first and second annuities be denoted by $F$ and $S$. Then $F = v^{-1}S + S$ from which $v^{-1} = (F - S)/S$. Also $s_{\overline{3}|}X = F$. Now the first fact gives $i = 0.10$, and the second gives $X = 9.996/3.31 = 3.02$. **B**.

**5**. The first two facts give $P = 1000ra_{\overline{10}|.04} + 1100(1.04)^{-10}$ and $P - 81.49 = 1000ra_{\overline{10}|.05} + 1100(1.05)^{-10}$. Subtracting the second from the first and solving gives $r = 0.035$. So the price to yield $r$ is $1000ra_{\overline{10}|r} + 1100(1+r)^{-10} = 1070.89$. **D**.

**6**. The equation of value is $1000 = 500v^{4/12} + 200v^{14/12} + Rv^{18/12}$ giving $R = 406.91$. **C**.

**7**. The given facts yield $100e^{\int_0^5 kt\,dt} = 250$, from which $25k/2 = \ln(2.5)$ or $k = 0.08\ln(2.5)$. **C**.

**8**. Since $a_{\overline{3n}|} = a_{\overline{n}|}(1+v^n+v^{2n})$, $v^n = 0.8$, and so $a_{\overline{4n}|} = a_{\overline{n}|} + v^n a_{\overline{3n}|} = 10 + (0.8)(24.40) = 29.52$. **D**.

## §16. Practice Examination 6

**1**. At an annual effective interest rate of $i > 0$, the following are all equal

(1) the present value of 10000 at the end of 6 years

(2) the sum of the present values of 6000 at the end of year $t$ and 56000 at the end of year $2t$

(3) 5000 immediately

Calculate the present value of a payment of 8000 at the end of year $t + 3$ using the same annual effective interest rate.

**A.** 1330

**B.** 1415

**C.** 1600

**D.** 1775

**E.** 2000

**2**. Jim buys a 10 year bond with par value of 10000 and 8% semiannual coupons. The redemption value of the bond at the end of 10 years is 10500. Calculate the purchase price to yield 6% convertible quarterly.

**A.** 11700

**B.** 14100

**C.** 14600

**D.** 15400

**E.** 17700

**3**. Barbara purchases an increasing perpetuity with payments occurring at the end of every 2 years. The first payment is 1, the second one is 2, the third one is 3, etc. The price of the perpetuity is 110. Calculate the annual effective interest rate.

**A.** 4.50%

**B.** 4.62%

**C.** 4.75%

**D.** 4.88%

**E.** 5.00%

**4**. Eric receives 12000 from a life insurance policy. He uses the fund to purchase two different annuities, each costing 6000. The first annuity is a 24 year anuity immediate paying $K$ per year to himself. The second annuity is an 8 year annuity immediate paying $2K$ per year to his son. Both annuities are based on an annual effective interest rate of $i > 0$. Determine $i$.

**A.** 6.0%

**B.** 6.2%

**C.** 6.4%

**D.** 6.6%

**E.** 6.8%

**5**. Victor wants to purchase a perpetuity paying 100 per year with the first payment due at the end of year 11. He can purchase it by either

(1) paying 90 per year at the end of each year for 10 years, or

(2) paying $K$ per year at the end of each year for the first 5 years and nothing for the next 5 years.

Calculate $K$.

**A.** 150

**B.** 160

**C.** 170

**D.** 175

**E.** 180

**6**. Esther invests 100 at the end of each year for 12 years at an annual effective interest rate of $i$. The interest payments are reinvested at an annual effective rate of 5%. The accumulated value at the end of 12 years is 1748.40. Calculate $i$.

**A.** 6%

**B.** 7%

**C.** 8%

**D.** 9%

**E.** 10%

**7**. At time $t = 0$ Billy puts 625 into an account paying 6% simple interest. At the end of year 2, George puts 400 into an account paying interest at a force of interest $\delta_t = \dfrac{1}{6+t}$, for $t \geq 2$. If both accounts continue to earn interest indefinitely at the levels given above, the amounts in both accounts will be equal at the end of year $n$. Calculate $n$.

**A.** 23

**B.** 24

**C.** 25

**D.** 26

**E.** 27

**8**. A bond with par value of 1000 and 6% semiannual coupons is redeemable for 1100. You are given

(1) the bond is purchased at $P$ to yield 8% convertible semiannually and

(2) the amount of principal adjustment for the 16th semiannual period is 5

Calculate $P$.

**A.** 760

**B.** 770

**C.** 790

**D.** 800

**E.** 820

## Solutions to Practice Examination 6

**1**. The information gives $5000 = 10000v^6$ and $5000 = 6000v^t + 56000v^{2t}$. The first gives $v^3 = \sqrt{1/2}$ while the second gives $v^t = 0.25$. The desired value is $8000v^{t+3} = 8000(0.25)\sqrt{1/2} = 1414.21$. **B**.

**2**. An interest rate of 6% convertible quarterly is the same as a rate of $i$ compounded semiannually where $(1 + .06/4)^4 = (1 + i/2)^2$, so $i = 0.06045$. The price of the bond is $400a_{\overline{20}|.06045/2} + 10500(1 + .06/4)^{-40} = 11726.88$. **A**.

**3**. From first principles, $\sum_{j=1}^{\infty} jv^{2j} = v^2/(v^2-1)^2$. Setting this equal to 110 and solving gives $v^2 = \sqrt{10/11}$ and $i = 0.0488$. **D**.

**4**. The information gives $6000 = Ka_{\overline{24}|}$ and $6000 = 2Ka_{\overline{8}|}$. Since $a_{\overline{24}|} = a_{\overline{8}|}(1+v^8+v^{16})$ taking the ratio of the the two original equations gives $1 = (1+v^8+v^{16})/2$, a quadratic in $v^8$. Solving gives $v^8 = 0.618$, from which $i = 0.0620$. **B**.

**5**. The information gives $90a_{\overline{10}|} = Ka_{\overline{5}|} = 100v^{10}/i$, where the last expression is the value of the perpetuity. Multiplying by $i$ gives $90(1 - v^{10}) = K(1 - v^5) = 100v^{10}$. Equating the extreme members gives $v^{10} = 90/190$. Hence $K = 151.94$. **A**.

**6**. The equation of value is $1200 + 100i(Is)_{\overline{11}|.05} = 1748.40$. Since $(Is)_{\overline{11}|.05} = 78.34$, solving gives $i = 0.070$. **B**.

**7**. The amount in Billy's account at time $n$ is $625(1 + .06n)$ while the amount in George's account is $400e^{\int_2^n \delta_t\,dt} = 400(n + 6)/8$. Equating gives $n = 26$. **D**.

**8**. The given information implies $P = 1100 + (30-44)a_{\overline{2n}|.04}$, from which $P < 1100$, and the periodic amortization is $44 - 30 = 14$. Also $14 - 14(.04)a_{\overline{2n-15}|.04} = 5$, so $a_{\overline{2n-15}|.04} = 16.071$, from which $2n - 15 = 27$ or $2n = 42$. Thus $P = 817.40$. **E**.

## §17. Practice Examination 7

**1**. A perpetuity with annual payments is payable beginning 10 years from now. The first payment is 50. Each annual payment thereafter is increased by 10 until a payment of 150 is reached. Subsequent payments remain level at 150. This perpetuity is purchased by means of 10 annual premiums, with the first premium of $P$ due immediately. Each premium after the first is 105% of the preceding one. The annual effective interest rates are 5% during the first 9 years and 3% thereafter. Calculate $P$.

**A.** 281

**B.** 286

**C.** 291

**D.** 296

**E.** 301

**2**. A deposit of 1 is made at the end of each year for 30 years into a bank account that pays interest at the end of each year at $j$ per annum. Each interest payment is reinvested to earn an annual effective interest rate of $j/2$. The accumulated value of these interest payments at the end of 30 years is 72.88. Determine $j$.

**A.** 8.0%

**B.** 8.5%

**C.** 9.0%

**D.** 9.5%

**E.** 10.0%

**3**. An investment fund is established at time 0 with a deposit of 5000. 1000 is added at the end of 4 months, and an additional 4000 is added at the end of 8 months. No withdrawals are made. The fund value, including interest, is 10560 at the end of 1 year. The force of interest at time $t$ is $\dfrac{k}{1+(1-t)k}$ for $0 \le t \le 1$. Determine $k$.

**A.** 0.072

**B.** 0.076

**C.** 0.080

**D.** 0.084

**E.** 0.088

**4**. Bart buys a 28 year bond with a par value of 1200 and annual coupons. The bond is redeemable at par. Bart pays 1968 for the bond, assuming an annual effective yield rate of $i$. The coupon rate on the bond is twice the yield rate. At the end of 7 years Bart sells the bond for $P$, which produces the same annual effective yield rate of $i$ to the new buyer. Calculate $P$.

**A.** 1470

**B.** 1620

**C.** 1680

**D.** 1840

**E.** 1880

**5**. An investment will triple in 87.88 years at a constant force of interest $\delta$. Another investment will quadruple in $t$ years at a nominal rate of interest numerically equal to $\delta$ and convertible once every 4 years. Calculate $t$.

**A.** 101 years

**B.** 106 years

**C.** 109 years

**D.** 114 years

**E.** 125 years

**6**. John borrows 10000 for 10 years and uses a sinking fund to repay the principal. The sinking fund deposits earn an annual effective interest rate of 5%. The total required payment for both the interest and the sinking fund deposit made at the end of each year is 1445.04. Calculate the annual effective interest rate charged on the loan.

**A.** 5.5%

**B.** 6.0%

**C.** 6.5%

**D.** 7.0%

**E.** 7.5%

**7**. You are given $(Ia)_{\overline{n-1}|} = K/d$, where $d$ is the annual effective discount rate. Calculate $K$.

**A.** $\ddot{a}_{\overline{n-1}|} - (n-1)v^{n-1}$

**B.** $a_{\overline{n-1}|} - nv^n$

**C.** $\ddot{a}_{\overline{n}|} - (n-1)v^n$

**D.** $a_{\overline{n}|} - (n-1)v^{n-1}$

**E.** $a_{\overline{n}|} - nv^n$

**8**. A 10 year bond with coupons at 8% convertible quarterly will be redeemed at 1600. The bond is bought to yield 12% convertible quarterly. The purchase price is 860.40. Calculate the par value.

**A.** 800

**B.** 1000

**C.** 1200

**D.** 1400

**E.** 1600

## Solutions to Practice Examination 7

**1**. Here

$$P\sum_{j=0}^{9}(1.05)^j(1.05)^{-j} = (1.05)^{-9}(50/.03 + (1.03)^{-1}10(Ia)_{\overline{9}|.03} + (1.03)^{-10}100/.03.$$

Thus $P = 290.72$. **C**.

**2**. The equation is $j(Is)_{\overline{29}|j/2} = 72.88$. Now $(Is)_{\overline{29}|j/2} = 2(\ddot{s}_{\overline{29}|j/2}-29)/j$, so the equation to solve is $\ddot{s}_{\overline{29}|j/2} = 65.44$, from which $j/2 = 0.05$. **E**.

**3**. Here $\int_x^1 \delta_t\,dt = \ln((1-x)k+1)$. Thus $5000(k+1)+1000((2/3)k+1)+4000((1/3)k+1) = 10560$, and $k = 0.08$. **C**.

**4**. The information gives $1968 = 1200 + (2400i - 1200i)a_{\overline{28}|i}$ and $P = 1200 + (2400i - 1200i)a_{\overline{21}|i}$. The first equation gives $1200(1 - v^{28}) = 1968 - 1200$, from which $i = 0.03716$ and $P = 1842.29$. **D**.

**5**. The information means $e^{87.88\delta} = 3$ and $(1 + 4\delta)^{t/4} = 4$. Thus $t = 4\ln(4)/\ln(1 + 4\delta) = 4\ln(4)/\ln(1 + 4\ln(3)/87.88) = 113.64$. **D**.

**6**. Here $1445.04 = 10000i + 10000/s_{\overline{10}|.05}$, from which $i = 0.065$. **C**.

**7**. Since $(Ia)_{\overline{n-1}|} = (a_{\overline{n-1}|} - (n-1)v^n)/d$ the numerator of this fraction is $K$. But $K = a_{\overline{n-1}|} - (n-1)v^n = v + \ldots + v^{n-1} - (n-1)v^n = v + \ldots + v^n - nv^n = a_{\overline{n}|} - nv^n$. **E**.

**8**. Here $860.40 = 1600 + (.02P - .03(1600))a_{\overline{40}|.03}$, from which $P = 800.15$. **A**.

# §18. Practice Examination 8

**1**. You are given

(1) the present value of an annuity due that pays 300 every 6 months during the first 15 years and 200 every 6 months during the second 15 years is 6000

(2) the present value of a 15 year deferred annuity due that pays 350 every 6 months for 15 years is 4000

(3) the present value of an annuity due that pays 100 every 6 months during the first 15 years and 200 every 6 months during the next 15 years is $X$.

(4) the same interest rate is used in all calculations

Determine $X$.

**A.** 3220

**B.** 3320

**C.** 3420

**D.** 3520

**E.** 3620

**2**. At time 0, 100 is deposited into Fund X and also into Fund Y. Fund X accumulates at a force of interest $\delta_t = 0.5(1+t)^{-2}$. Fund Y accumulates at an annual effective interest rate of $i$. At the end of 9 years, the accumulated value of Fund X equals the accumulated value of Fund Y. Determine $i$.

**A.** 4.5%

**B.** 4.8%

**C.** 5.1%

**D.** 5.4%

**E.** 5.7%

**3**. John invests 1000 in a fund which earns interest during the first year at a nominal rate of $K$ convertible quarterly. During the 2nd year the fund earns interest at a nominal discount rate of $K$ convertible quarterly. At the end of the 2nd year, the fund has accumulated to 1173.54. Calculuate $K$.

**A.** 0.064

**B.** 0.068

**C.** 0.072

**D.** 0.076

**E.** 0.080

**4**. A bank agrees to lend John 10,000 now and $X$ three years later in exchange for a single repayment of 75,000 at the end of 10 years. The bank charges interest at an annual effective rate of 6% for the first 5 years and at a force of interest $\delta_t = 1/(t+1)$ for $t \geq 5$. Determine $X$.

**A.** 23,500

**B.** 24,000

**C.** 24,500

**D.** 25,000

**E.** 25,500

**5**. Dottie receives payments of $X$ at the end of each year for $n$ years. The present value of her annuity is 493. Sam receives payments of $3X$ at the end of each year for $2n$ years. The present value of his annuity is 2748. Both present values are calculated at the same annual effective interest rate. Determine $v^n$.

**A.** 0.86

**B.** 0.87

**C.** 0.88

**D.** 0.89

**E.** 0.90

**6**. Jeff deposits 100 at the end of each year for 13 years into Fund X. Antoinette deposits 100 at the end of each year for 13 years into Fund Y. Fund X earns an annual effective rate of 15% for the first 5 years and an annual effective rate of 6% thereafter. Fund Y earns an annual effective rate of $i$. At the end of 13 years, the accumulated value of Fund X equals the accumulated value of Fund Y. Calculate $i$.

**A.** 6.4%

**B.** 6.7%

**C.** 7.0%

**D.** 7.4%

**E.** 7.8%

**7**. Francois purchases a 10 year annuity immediate with annual payments of $10X$. Jacques purchases a 10 year decreasing annuity immediate which also makes annual payments. The payment at the end of year 1 is equal to 50. At the end of year 2, and at the end of each year through year 10, each subsequent payment is reduced over what was paid in the previous year by an amount equal to $X$. At an annual effective interest rate of 7.072%, both annuities have the same present value. Calculate $X$, where $X < 5$.

**A.** 3.29

**B.** 3.39

**C.** 3.49

**D.** 3.59

**E.** 3.69

**8**. A 5000 serial bond with 10% annual coupons will be redeemed in five equal installments of 1100 beginning at the end of the 11th year and continuing through the 15th year. The bond was bought at a price $P$ to yield 9% annual effective. Determine $P$.

**A.** 5000

**B.** 5227

**C.** 5537

**D.** 5764

**E.** 5991

## Solutions to Practice Examination 8

**1**. The information yields $300\ddot{a}_{\overline{30}|} + 200v^{30}\ddot{a}_{\overline{30}|} = 6000$, $350v^{30}\ddot{a}_{\overline{30}|} = 4000$, and $100\ddot{a}_{\overline{30}|} + 200v^{30}\ddot{a}_{\overline{30}|} = X$. The second gives $v^{30}\ddot{a}_{\overline{30}|} = 4000/350 = 11.428$, and using this in the first gives $\ddot{a}_{\overline{30}|} = (6000 - 200(4000/350))/300 = 12.38$. Plugging in these values gives $X = 3523.69$. **D**.

**2**. The equation of value is $e^{\int_0^9 0.5(1+t)^{-2}\,dt} = (1+i)^9$, from which $e^{0.5-0.05} = (1+i)^9$ and $i = 0.0512$. **C**.

**3**. Here $1000(1+K/4)^4(1-K/4)^{-4} = 1173.54$, from which $(1+K/4)/(1-K/4) = (1.17354)^{1/4}$ and $K = 0.08$. **E**.

**4**. At time 5 $10000(1.06)^5 + X(1.06)^2 = 75000e^{-\int_5^{10} 1/(t+1)\,dt} = 75000(6/11)$, from which $X = 24,498.78$. **C**.

**5**. Since $Xa_{\overline{n}|} = 493$ and $3Xa_{\overline{2n}|} = 2748$, using the fact that $a_{\overline{2n}|} = a_{\overline{n}|}(1+v^n)$ in the second fact and substituting gives $3(493)(1+v^n) = 2748$, from which $v^n = 0.858$. **A**.

**6**. Here $100((1.06)^8 s_{\overline{5}|.15} + s_{\overline{8}|.06}) = 100s_{\overline{13}|i}$, from which $s_{\overline{13}|i} = 20.643$ and $i = 0.0738$. **D**.

**7**. Here $(50 - 10X)a_{\overline{10}|} + X(Da)_{\overline{10}|} = 10Xa_{\overline{10}|}$, from which $X = 3.586$. **D**.

**8**. Here $P = 500a_{\overline{10}|.09} + (1.09)^{-10}100(Da)_{\overline{5}|.09} + (1.09)^{10}1100a_{\overline{5}|.09}$, giving $P = 5537.29$. **C**.

## §19. Practice Examination 9

**1**. Jeff bought an increasing perpetuity due with annual payments starting at 5 and increasing by 5 each year until the payment reaches 100. The payments remain at 100 thereafter. The annual effective interest rate is 7.5%. Determine the present value of this perpetuity.

**A.** 700

**B.** 735

**C.** 760

**D.** 785

**E.** 810

**2**. On January 1, 1999, Luciano deposits 90 into an investment account. On April 1, 1999, when the amount in Luciano's account is equal to $X$, a withdrawal of $W$ is made. No further deposits or withdrawals are made to Luciano's account for the remainder of the year. On December 31, 1999, the amount in Luciano's account is 85. The dollar weighted return over the 1 year period is 20%. The time weighted return over the 1 year period is 16%. Calculate $X$.

**A.** 101.6

**B.** 103.6

**C.** 105.6

**D.** 107.6

**E.** 109.6

**3**. Coco invests 2000 at the beginning of the year in a fund which credits interest at an annual effective rate of 9%. Coco reinvests each interest payment in a separate fund, accumulating at an annual effective rate of 8%. The interest payments from this fund accumulate in a bank account that guarantees an annual effective rate of 7%. Determine the sum of the principal and interest at the end of 10 years.

**A.** 4505

**B.** 4545

**C.** 4585

**D.** 4625

**E.** 4665

**4**. Jim borrowed 10,000 from Bank X at an annual effective rate of 8%. He agreed to repay the bank with five level annual installments at the end of each year. At the same time, he also borrowed 15,000 from Bank Y at an annual effective rate of 7.5%. He agreed to repay this loan with five level annual installments at the end of each year. He lent the 25,000 to Wayne immediately in exchange for four annual level repayments at the end of each year, at an annual effective rate of 8.5%. Jim can only reinvest the proceeds at an annual effective rate of 6%. Immediately afer repaying the loans to the banks in full, determine how much Jim has left.

**A.** 323

**B.** 348

**C.** 373

**D.** 398

**E.** 423

**5**. Raj deposits 100 into a fund at the end of each 2 year period for 20 years. The fund pays interest at an annual effective rate of $i$. The total amount of interest earned by the fund during the 19th and 20th years is 250. Calculate the accumulated amount in Raj's account at the end of year 20.

**A.** 1925

**B.** 1950

**C.** 1975

**D.** 2000

**E.** 2025

**6**. Nikita takes out a 10 year loan. The loan is repaid by making 10 annual repayments at the end of each year. The first loan repayment is equal to $X$, with each subsequent repayment 10.16% greater than the previous year's repayment. The annual effective interest rate being charged on the loan is 8%. The amount of interest repaid during the first year is equal to 892.20. Calculate $X$.

**A.** 1100

**B.** 1150

**C.** 1200

**D.** 1250

**E.** 1300

**7**. A 1000 par value 10 year bond with semiannual coupons and redeemable at 1100 is purchased at 1135 to yield 12% convertible semiannually. The first coupon is $X$. Each subsequent coupon is 4% greater than the preceding coupon. Determine $X$.

**A.** 40

**B.** 42

**C.** 44

**D.** 48

**E.** 50

**8**. Ming borrows $X$ for 10 years at an annual effective interest rate of 8%. If he pays the principal and accumulated interest in one lump sum at the end of 10 years, he would pay 468.05 more in interest than if he repaid the loan with 10 level payments at the end of each year. Calculate $X$.

**A.** 675

**B.** 700

**C.** 725

**D.** 750

**E.** 775

## Solutions to Practice Examination 9

**1**. The present value is $5/d + 5(Ia)_{\overline{19}|} + v^{19}95/i = 785.40$. **D**.

**2**. The information given is that $(X/90)(85/(X-W)) = 1.16$ and $90(1.2)-W(1.2)^{3/4} = 85$. The second gives $W = 20.06$, and thus $X = 107.95$. **D**

**3**. Here 180 flows from the the first fund at the end of each year into the second fund. Starting at the end of the second year, contributions starting at 14.40 and increasing by a like amount annually begin flowing into the bank. So the amount at the end of the 10 years is $2000 + 10(180) + 14.40(Is)_{\overline{9}|.07} = 4585.09$. **C**

**4**. Jim's 5 payments are $10000/a_{\overline{5}|.08} = 2504.56$ and $15000/a_{\overline{5}|.075} = 3707.47$ for a total of 6212.03 while Wayne's 4 payments are $25000/a_{\overline{4}|.085} = 7632.20$. So the amount Jim has is $(1.06)(7632.20 - 6212.03)s_{\overline{4}|.06} - 6212.03 = 373.43$. **C**.

**5**. The amount in the fund at the end of the $k$th 2 year period is $100\sum_{j=1}^{k}(1+i)^{2k-2j} = 100((1+i)^{2k}-1)/((1+i)^2-1)$. Thus $100((1+i)^{18}-1) = 250$ and $i = 0.072$ and the accumulated value is 2022.29. **E**.

**6**. The amount of the loan is $892.20/0.08 = 11152.50$. Thus

$$11152.50 = X\sum_{j=1}^{10}(1.1016)^{j-1}v^j = Xv(1-((1.1016)v)^{10})/(1-1.1016v),$$

so $X = 1100$. **A**.

**7**. Here $1135 = X\sum_{j=1}^{20}(1.04)^{j-1}(1.06)^{-j} + (1.06)^{-20}1100$, from which $X = 50$. **E**.

**8**. From the information given, $X(1.08)^{10} - X = 468.05 + 10X/a_{\overline{10}|.08} - X$, from which $X = 700.01$. **B**.

## §20. Practice Examination 10

**1**. Eloise plans to accumulate 100,000 at the end of 42 years. She makes the following deposits:

(1) $X$ at the beginning of years 1–14;

(2) No deposits at the beginning of years 15–32; and

(3) $Y$ at the beginning of years 33–42.

The annual effective interest rate is 7%. $X - Y = 100$. Calculate $Y$.

**A.** 479

**B.** 499

**C.** 519

**D.** 539

**E.** 559

**2**. John borrows 10,000 for 10 years at an annual effective interest rate of $i$. He accumulates the amount necessary to repay the loan by using a sinking fund. He makes 10 payments of $X$ at the end of each year, which includes interest on the loan and the payment into the sinking fund, which earns an annual effective rate of 8%. If the annual effective rate of the loan had been $2i$, his total annual payment would have been $1.5X$. Calculate $i$.

**A.** 6.7%

**B.** 6.9%

**C.** 7.1%

**D.** 7.3%

**E.** 7.5%

**3**. The following table shows the annual effective interest rates being credited by an investment account, by calendar year of investment. The investment year method is applicable for the first 3 years, after which a portfolio rate is used.

| Calendar Year of Investment | Investment Year Rates $i_1$ | $i_2$ | $i_3$ | Calendar Year of Portfolio Rate | Portfolio Rate |
|---|---|---|---|---|---|
| 1990 | 10% | 10% | $t$% | 1993 | 8% |
| 1991 | 12% | 5% | 10% | 1994 | $(t-1)$% |
| 1992 | 8% | $(t-2)$% | 12% | 1995 | 6% |
| 1993 | 9% | 11% | 6% | 1996 | 9% |
| 1994 | 7% | 7% | 10% | 1997 | 10% |

An investment of 100 is made at the beginning of years 1990, 1991, and 1992. The total amount of interest credited by the fund during the year 1993 is equal to 28.40. Calculate $t$.

**A.** 7.00

**B.** 7.25

**C.** 7.50

**D.** 7.75

**E.** 8.00

**4**. Victor invests 300 into a bank account at the beginning of each year for 20 years. The account pays out interest at the end of every year at an annual effective interest rate of $i$%. The interest is reinvested at an annual effective rate of $(i/2)$%. The yield rate on the entire investment over the 20 year period is 8% annual effective. Determine $i$.

**A.** 9%

**B.** 10%

**C.** 11%

**D.** 12%

**E.** 13%

**5**. Kevin takes out a 10 year loan of $L$, which he repays by the amortization method at an annual effective interest rate of $i$. Kevin makes payments of 1000 at the end of each year. The total amount of interest repaid during the life of the loan is also equal to $L$. Calculate the amount of interest repaid during the first year of the loan.

**A.** 725

**B.** 750

**C.** 755

**D.** 760

**E.** 765

**6**. Sandy purchases a perpetuity immediate that makes annual payments. The first payment is 100, and each payment thereafter increases by 10. Danny purchases a perpetuity due which makes annual payments of 180. Using the same annual effective interest rate, $i > 0$, the present value of both perpetuities are equal. Calculate $i$.

**A.** 9.2%

**B.** 9.7%

**C.** 10.2%

**D.** 10.7%

**E.** 11.2%

**7**. Jerry will make deposits of 450 at the end of each quarter for 10 years. At the end of 15 years, Jerry will use the fund to make annual payments of $Y$ at the beginning of each year for 4 years, after which the fund is exhausted. The annual effective rate of interest is 7%. Determine $Y$.

**A.** 9573

**B.** 9673

**C.** 9773

**D.** 9873

**E.** 9973

**8**. Bill and Jane each sell a different stock short for a price of 1000. For both investors, the margin requirement is 50%, and interest on the margin is credited at an annual effective rate of 6%. Bill buys back his stock one year later at a price of $P$. A the end of the year, the stock paid a dividend of $X$. Jane also buys back her stock after one year, at a price of $(P - 25)$. At the end of the year her stock paid a dividend of $2X$. Both investors earned an annual effective yield of 21% on their short sales. Calculate $P$.

**A.** 800

**B.** 825

**C.** 850

**D.** 875

**E.** 900

## Solutions to Practice Examination 10

**1**. The equation of value is $(Y + 100)\ddot{s}_{\overline{14}|}(1.07)^{28} + Y\ddot{s}_{\overline{10}|} = 100000$, from which $Y = 479.17$. **A**.

**2**. From the given information, $X = 10000i + 10000/s_{\overline{10}|.08}$ and $1.5X = 10000(2i) + 10000/s_{\overline{10}|.08}$. Substituting the first into the second and computing gives $i = 1/s_{\overline{10}|.08} = 0.0690$. **B**.

**3**. The interest earned satisfies

$$28.40 = 100(1.1)(1.1)(1 + t/100)(0.08) + 100(1.12)(1.05)(0.10) + 100(1.08)((t-2)/100),$$

from which $t = 7.749$. **D**.

**4**. The interest earned in year $k$ on the first bank account is $300ik$, so the deposits in the second account form an increasing annuity which accumulates at rate $i/2$. Thus $300(20) + 300i(Is)_{\overline{20}|i/2} = 300\ddot{s}_{\overline{20}|0.08}$, from which $i = 0.0999$. **B**.

**5**. From the information given, $1000 = L/a_{\overline{10}|i}$ and $10000 - L = L$. Thus $L = 5000$, and the first fact gives $i = 0.1509$, so that the amount of interest in the first payment is $5000i = 754.92$. **C**.

**6**. The equation of value is $90/i + 10(Ia)_{\overline{\infty}|} = 180/d$, from which $i = 10.17\%$. **C**.

**7**. The equation of value is $450s_{\overline{40}|0.07^{(4)}/4}(1.07)^5 = Y\ddot{a}_{\overline{4}|0.07}$. This gives $Y = 9873.20$. **D**.

**8**. The information for Bill gives $(1000 - P + 30 - X)/500 = 0.21$ and for Jane gives $(1025 - P + 30 - 2X)/500 = 0.21$, from which $P = 900$ and $X = 25$. **E**.

## §21. Practice Examination 11

**1**. An investor deposits 50 in an investment account on January 1. The following summarizes the activity in the account during the year.

| Date | Value Immediately Before Deposit | Deposit |
|---|---|---|
| March 15 | 40 | 20 |
| June 1 | 80 | 80 |
| October 1 | 175 | 75 |

On June 30, the value of the account is 157.50. On December 31, the value of the account is $X$. Using the time weighted method, the equivalent annual effective yield during the first 6 months is equal to the time weighted annual effective yield during the entire 1 year period. Calculate $X$.

**A.** 234.75

**B.** 235.50

**C.** 236.25

**D.** 237.00

**E.** 237.75

**2**. A 1000 par value 20 year bond with annual coupons and redeemable at maturity at 1050 is purchased for $P$ to yield an annual effective rate of 8.25%. The first coupon is 75. Each subsequent coupon is 3% greater than the preceding coupon. Determine $P$.

**A.** 985

**B.** 1000

**C.** 1050

**D.** 1075

**E.** 1115

**3**. An investor took out a loan of 150,000 at 8% compounded quarterly, to be repaid over 10 years with quarterly payments of 5483.36 at the end of each quarter. After 12 payments the interest rate dropped to 6% compounded quarterly. The new quarterly payment dropped to 5134.62. After 20 payments in total, the interest rate on the loan increased to 7% compounded quarterly. The investor decided to make an additional payment of $X$ at the time of his 20th payment. After the additional payment was made, the new quarterly payment was calculated to be 4265.73, payable for five more years. Determine $X$.

**A.** 11,047

**B.** 13,369

**C.** 16,691

**D.** 20,152

**E.** 23,614

**4**. Chuck needs to purchase an item in 10 years. The item costs 200 today, but its price inflates 4% per year. To finance the purchase, Chuck deposits 20 into an account at the beginning of each year for 6 years. He deposits an additional $X$ at the beginning of years 4, 5, and 6 to meet his goal. The annual effective interest rate is 10%. Calculate $X$.

**A.** 7.4

**B.** 7.9

**C.** 8.4

**D.** 8.9

**E.** 9.4

**5**. Among a company's assets and accounting records, an actuary finds a 15 year bond that was purchased at a premium. From the records, the actuary has determined that the bond pays semi-annual interest, the amount for amortization of the premium in the second coupon payment was 977.19, and the amount for amortization of the premium in the fourth coupon payment was 1046.79. What is the value of the premium?

**A.** 17,365

**B.** 24,784

**C.** 26,549

**D.** 48,739

**E.** 50,445

**6**. Joe can purchase one of two annuities. Annuity 1 is a 10 year decreasing annuity immediate with annual payments of 10, 9, 8, . . . , 1. Annuity 2 is a perpetuity immediate with annual payments of 1 in year 1, 2 in year 2, . . . , and 11 in year 11. After year 11 the payments remain constant at 11. At an annual effective interest rate of $i$ the present value of Annuity 2 is twice the present value of Annuity 1. Calculate the value of Annuity 1.

**A.** 36.4

**B.** 37.4

**C.** 38.4

**D.** 39.4

**E.** 40.4

**7**. A 12 year loan of 8000 is to be repaid with payments to the lender of 800 at the end of each year and deposits of $X$ at the end of each year into a sinking fund. Interest on the loan is charged at an 8% annual effective rate. The sinking fund annual effective interest rate is 4%. Calculate $X$.

**A.** 298

**B.** 330

**C.** 361

**D.** 385

**E.** 411

**8**. At time 0, $K$ is deposited into Fund X, which accumulates at a force of interest $\delta_t = 0.006t^2$. At time $m$, $2K$ is deposited into Fund Y, which accumulates at an annual effective interest rate of 10%. At time $n$, where $n > m$, the accumulated value of each fund is $4K$. Determine $m$.

**A.** 1.6

**B.** 2.4

**C.** 3.8

**D.** 5.0

**E.** 6.2

## Solutions to Practice Examination 11

**1**. The time weighted yield for the first six months is

$$1 + i = (40/50)(80/60)(157.50/160) = 1.05$$

while the time weighted yield for the year is

$$1.05(175/157.50)(X/250) = 0.004666X.$$

Thus $(1.05)^2 = 0.004666X$ and $X = 236.25$. **C**.

**2**. Here $P = \sum_{j=1}^{20} 75(1.03)^{j-1}v^j + 1050v^{20} = 1115.11$. **E**.

**3**. The loan balance after 20 payments is $5134.62a_{\overline{20}|.015} = 88154.43$. Thus $88154.43 - X = 4265.73a_{\overline{20}|.07/4}$, so $X = 16691.16$. **C**.

**4**. The equation of value is $200(1.04)^{10} = 20\ddot{s}_{\overline{6}|.1}(1.1)^4 + X\ddot{s}_{\overline{3}|.1}(1.1)^4$, from which $X = 8.915$. **D**.

**5**. From the given information $v^2 = 977.19/1046.79$. The total premium is $pa_{\overline{30}|}$ where $pv^{29} = 977.19$. Putting these together gives $i = 0.035$ per half year, $p = 2650$ and the total premium as 48739. **D**.

**6**. The equation of value is $2(Da)_{\overline{10}|} = (Ia)_{\overline{11}|} + v^{11}11/i$, from which $i = 0.0798$, and the value of Annuity 1 is 39.42. **D**.

**7**. The equation of value is $8000(1.08)^{12} = 800s_{\overline{12}|.08} + Xs_{\overline{12}|.04}$. Thus $X = 330.34$. **B**.

**8**. For Fund X the amount at time $n$ is $K\exp\{\int_0^n .006t^2\,dt\} = Ke^{.002n^3}$. The amount in Fund Y at time $n > m$ is $2K(1.1)^{n-m}$. Setting the first equal to $4K$ gives $n = 8.85$, and then setting the second equal to $4K$ gives $m = 1.57$. **A**.

## §22. Practice Examination 12

**1**. Iggy borrows $X$ for 10 years at an annual effective rate of 6%. If he pays the principal and accumulated interest in one lump sum at the end of 10 years, he would pay 356.54 more in interest than if he repaid the loan with 10 level payments at the end of each year. Calculate $X$.

**A.** 800

**B.** 825

**C.** 850

**D.** 875

**E.** 900

**2**. A 20 year loan of 20,000 may be repaid under the following two methods.

(1) Method 1 amortizes the loan with equal annual payments at an effective rate of 6.5%.

(2) Method 2 is a sinking fund method in which the lender receives an annual effective rate of 8% and the sinking fund earns an annual effective rate of $j$.

Both methods require a payment of $X$ to be made at the end of each year for 20 years. Calculate $j$.

**A.** $j \le 6.5\%$

**B.** $6.5\% < j \le 8.0\%$

**C.** $8.0\% < j \le 10.0\%$

**D.** $10.0\% < j \le 12.0\%$

**E.** $j > 12.0\%$

**3**. A perpetuity immediate pays $X$ per year. Brian receives the first $n$ payments, Colleen receives the next $n$ payments, and Jeff receives the remaining payments. Brian's share of the present value of the original perpetuity is 40%, and Jeff's share is $K$. Calculate $K$.

**A.** 24%

**B.** 28%

**C.** 32%

**D.** 36%

**E.** 40%

**4**. Seth, Janice, and Lori each borrow 5000 for five years at a nominal interest rate of 12% compounded semi-annually. Seth has interest accumulated over the five years and pays all the interest and principal in a lump sum at the end of five years. Janice pays interest at the end of every six-month period as it accrues and the principal at end of the five years. Lori repays her loan with 10 level payments at the end of every six-month period. Calculate the total amount of interest paid on all three loans.

**A.** 8718

**B.** 8728

**C.** 8738

**D.** 8748

**E.** 8758

**5**. Bruce and Robbie each open up new bank accounts at time 0. Bruce deposits 100 into his bank account, and Robbie deposits 50 into his. Each account earns an annual effective discount rate of $d$. The amount of interest earned in Bruce's account during the 11th year is equal to $X$. The amount of interest earned in Robbie's account during the 17th year is also equal to $X$. Calculate $X$.

**A.** 28.0

**B.** 31.3

**C.** 34.6

**D.** 36.7

**E.** 38.9

**6**. Ron has a loan with a present value of $a_{\overline{n}|}$. The sum of the interest paid in period $t$ plus the principal repaid in period $t+1$ is $X$. Calculate $X$.

**A.** $1+\dfrac{v^{n-t}}{i}$

**B.** $1+\dfrac{v^{n-t}}{d}$

**C.** $1+v^{n-t}i$

**D.** $1+v^{n-t}d$

**E.** $1+v^{n-t}$

**7**. At an annual effective interest rate of $i$, $i > 0\%$, the present value of a perpetuity paying 10 at the end of each 3 year period, with the first payment at the end of year 6, is 32. At the same annual effective rate of $i$, the present value of a perpetuity immediate paying 1 at the end of each 4 month period is $X$. Calculate $X$.

**A.** 38.8

**B.** 39.8

**C.** 40.8

**D.** 41.8

**E.** 42.8

**8**. Susan invests $Z$ at the end of each year for seven years at an annual effective interest rate of 5%. The interest credited at the end of each year is reinvested at an annual effective rate of 6%. The accumulated value at the end of seven years is $X$. Lori invests $Z$ at the end of each year for 14 years at an annual effective interest rate of 2.5%. The interest credited at the end of each year is reinvested at an annual effective rate of 3%. The accumulated value at the end of 14 years is $Y$. Calculate $Y/X$.

**A.** 1.93

**B.** 1.98

**C.** 2.03

**D.** 2.08

**E.** 2.13

## Solutions to Practice Examination 12

**1**. The information gives $(X(1.06)^{10} - X) - (10X/a_{\overline{10}|.06} - X) = 356.64$, from which $X = 825.23$. **B**.

**2**. Method 1 gives $X = 20000/a_{\overline{20}|.065} = 1815.13$. The sinking fund contribution under Method 2 is $1815.13 - 1600 = 215.13$. Thus $215.13 s_{\overline{20}|j} = 20000$, which gives $j = 14.18\%$. **E**.

**3**. From the information about Brian's share, $Xa_{\overline{n}|} = 0.40X/i$, from which $1 - v^n = 0.40$ and $v^n = .60$. The information about Jeff's share is $v^{2n}X/i = KX/i$ so that $K = v^{2n} = 0.36$. **D**.

**4**. Seth's interest is $5000(1.06)^{10} - 5000 = 3954.24$, Janice's interest is $300 \times 10 = 3000$, and Lori's interest is $5000/a_{\overline{10}|.06} \times 10 - 5000 = 1793.40$. The total is 8747.64. **D**.

**5**. Bruce's account balance at the end of the 11th year is $100(1-d)^{-11}$, so the interest earned in the 11th year is $100d(1-d)^{-11}$. Similarly the interest earned in Robbie's account during the 17th year is $50d(1-d)^{-17}$. Thus $(1-d)^6 = 1/2$, $d = 0.109$ and $X = 38.79$. **E**.

**6**. The loan balance after the $t-1$st payment is $a_{\overline{n-t+1}|}$, so the interest paid in period $t$ is $1 - v^{n-t+1}$. The principal paid in period $t+1$ is $v^{n+1-(t+1)} = v^{n-t}$. Thus $X = 1 - v^{n-t+1} + v^{n-t} = 1 + v^{n-t}(1-v) = 1 + v^{n-t}d$. **D**.

**7**. From the first infomation, $32 = 10(v^6 + v^9 + \ldots) = 10v^6/(1-v^3)$. so that $i = 0.0772$. The equivalent interest rate per 4 month period is $i^{(3)}/3 = ((1.0772)^{1/3} - 1) = 0.02509$, giving the value of the second perpetuity as $1/.02509 = 39.84$. **B**.

**8**. Here $X = 7Z + 0.05Z(Is)_{\overline{6}|.06}$ and $Y = 14Z + 0.025Z(Is)_{\overline{13}|.03}$. Thus $Y/X = 2.03$. **C**.

# §23. Practice Examination 13

**1**. Jose and Chris each sell a different stock short for the same price. For each investor the margin requirement is 50% and interest on the margin debt is paid at an annual effective rate of 6%. Each investor buys back his stock one year later at a price of 760. Jose's stock paid a dividend of 32 at the end of the year while Chris's stock paid no dividends. During the 1 year period, Chris's return on the short sale is $i$, which is twice the return earned by Jose. Calculate $i$.

**A.** 12%

**B.** 16%

**C.** 18%

**D.** 20%

**E.** 24%

**2**. You are given the following information about an investment account.

| Date | Value Immediately Before Deposit | Deposit |
|---|---|---|
| January 1 | 10 | |
| July 1 | 12 | $X$ |
| December 31 | $X$ | |

Over the year, the time weighted return is 0% and the dollar weighted return is $Y$. Calculate $Y$.

**A.** −25%

**B.** −10%

**C.** 0%

**D.** 10%

**E.** 25%

**3**. Seth borrows $X$ for four years at an annual effective interest rate of 8%, to be repaid with equal payments at the end of each year. The outstanding loan balance at the end of the second year is 1076.82 and at the end of the third year is 559.12. Calculate the principal repaid in the first payment.

**A.** 444

**B.** 454

**C.** 464

**D.** 474

**E.** 484

**4**. Bill buys a 10 year 1000 par value 6% bond with semi-annual coupons. The price assumes a nominal yield of 6% compounded semi-annually. As Bill receives each coupon payment, he immediately puts the money into an account earning interest at an annual effective rate of $i$. At the end of 10 years, immediately after Bill receives the final coupon payment and the redemption value of the bond, Bill has earned an annual effective yield of 7% on his investment in the bond. Calculate $i$.

**A.** 9.50%

**B.** 9.75%

**C.** 10.00%

**D.** 10.25%

**E.** 10.50%

**5**. At time $t = 0$, 1 is deposited into each of Fund X and Fund Y. Fund X accumulates at a force of interest $\delta_t = \frac{t^2}{k}$. Fund Y accumulates at a nominal rate of discount of 8% per annum convertible semiannually. At time $t = 5$ the accumulated value of Fund X equals the accumulated value of Fund Y. Determine $k$.

**A.** 100

**B.** 102

**C.** 104

**D.** 106

**E.** 108

**6**. A company's stock is currently selling for 28.50. Its next dividend, payable one year from now, is expected to be 0.50 per share. Analysts forecast a long-run dividend growth rate of 7.5% for the company. Tomorrow the long-run dividend growth rate estimate changes to 7%. Calculate the new stock price.

**A.** 22.2

**B.** 23.8

**C.** 25.9

**D.** 28.0

**E.** 28.5

**7**. Tawny makes a deposit into a bank account which credits interest at a nominal interest rate of 10% per annum, convertible semiannually. At the same time, Fabio deposits 1000 into a different bank account, which is credited with simple interest. At the end of 5 years, the forces of interest on the two accounts are equal, and Fabio's account has accumulated to $Z$. Determine $Z$.

**A.** 1792

**B.** 1953

**C.** 2092

**D.** 2153

**E.** 2392

**8**. The present values of the following three annuities are equal.

(1) A perpetuity immediate paying 1 each year, calculated at an annual effective interest rate of 7.25%.

(2) A 50 year annuity immediate paying 1 each year, calculated at an annual effective interest rate of $j$%.

(3) An $n$ year annuity immediate paying 1 each year, calculated at an annual effective interest rate of $j - 1$%.

Calculate $n$.

**A.** 30

**B.** 33

**C.** 36

**D.** 39

**E.** 42

## Solutions to Practice Examination 13

**1**. Chris's return is $(0.06(P/2)+P-760)/(P/2)$ and Jose's return is $(0.06(P/2)+P-760-32)/(P/2)$ where $P$ is the sales price of the stock. Since Chris's return is twice Jose's, $1.03P-760 = 2(1.03P-792)$ and $P = 800$ and Chris's return is 16%. **B**.

**2**. The time weighted return is $(12/10)(X/(12+X)) = 1+0$ so $X = 60$. The dollar weighted return satisfies $10(1+Y)+60(1+Y)^{1/2} = 60$. Using the standard approximation, $10(1+Y)+60(1+Y/2) = 60$, from which $Y = -.25$. **A**.

**3**. Each payment is $X/a_{\overline{4}|.08}$, and $Xa_{\overline{2}|.08}/a_{\overline{4}|.08} = 1076.82$ so that $X = 2000$ and the annual payment is 603.84. Since the interest part of the first payment is 160, the principal part of the first payment is 443.84. **A**.

**4**. Bill paid 1000 for the bond. Thus $1000(1.07)^{10} = 1000+30s_{\overline{20}|i^{(2)}/2}$. Thus $i^{(2)}/2 = 0.04759$, and $i = 0.974$. **B**.

**5**. The given information implies $e^{125/3k} = (1-.08/2)^{-10}$, from which $k = 102.069$. **B**.

**6**. The current price was determined by the equation $28.50 = \sum_{j=1}^{\infty} 0.50(1.075)^{j-1}v^j$ for some $v$. Computing gives $v = 0.91529$. The new price is $\sum_{j=1}^{\infty} 0.050(1.07)^{j-1}v^j = 22.17$. **A**.

**7**. The amount in Tawny's account at time $t$ is $A(1.05)^{2t}$, so the force of interest on her account at time $t$ is $\frac{d}{dt}(1.05)^{2t}/(1.05)^{2t} = 0.09758$. Similarly, the amount in Fabio's account at time $t$ is $1000(1+it)$ and his force of interest is $i/(1+it)$. Equating the two forces at $t = 5$ gives $i = 0.19$ and the amount in his account is 1950. **B**.

**8**. The present values are 13.793, $a_{\overline{50}|j/100}$, and $a_{\overline{n}|(j-1)/100}$. Equating the first two gives $j = 7\%$. Equating the first and third gives $n = 30.17$. **A**.

## §24. Practice Examination 14

**1**. At time 0, deposits of 10,000 are made into each of Fund X and Fund Y. Fund X accumulates at an annual effective interest rate of 5%. Fund Y accumulates at a simple interest rate of 8%. At time $t$ the forces of interest on the two funds are equal. At time $t$ the accumulated value of Fund Y is greater than the accumulated value of Fund X by Z. Determine Z.

**A.** 1625

**B.** 1687

**C.** 1697

**D.** 1711

**E.** 1721

**2**. At a force of interest $\delta_t = \dfrac{2}{k+2t}$

(1) a deposit of 75 at time $t = 0$ will accumulate to $X$ at time $t = 3$; and

(2) the present value at time $t = 3$ of a deposit of 150 at time $t = 5$ is also equal to $X$.

Calculate $X$.

**A.** 105

**B.** 110

**C.** 115

**D.** 120

**E.** 125

**3**. Brian and Jennifer each take out a loan of $X$. Jennifer will repay her loan by making one payment of 800 at the end of year 10. Brian will repay his loan by making one payment of 1120 at the end of year 10. The nominal semi-annual rate being charged to Jennifer is exactly one-half the nominal semi-annual rate being charged to Brian. Calculate $X$.

**A.** 562

**B.** 565

**C.** 568

**D.** 571

**E.** 574

**4**. Carol and John shared equally in an inheritance. Using his inheritance John immediately bought a 10 year annuity due with an annual payment of 2500 each. Carol put her inheritance in an investment fund earning an annual effective interest rate of 9%. Two years later Carol bought a 15 year annuity immediate with annual payment of $Z$. The present value of both annuities was determined using an annual effective interest rate of 8%. Calculate $Z$.

**A.** 2330

**B.** 2470

**C.** 2515

**D.** 2565

**E.** 2715

**5**. Susan and Jeff each make deposits of 100 at the end of each year for 40 years. Starting at the end of the 41st year, Susan makes annual withdrawals of $X$ for 15 years and Jeff makes annual withdrawals of $Y$ for 15 years. Both funds have a balance of 0 after the last withdrawal. Susan's fund earns an annual effective interest rate of 8%. Jeff's fund earns an annual effective interest rate of 10%. Calculate $Y - X$.

**A.** 2792

**B.** 2824

**C.** 2859

**D.** 2893

**E.** 2925

**6**. A loan of 10,000 is to be amortized in 10 annual payments beginning 6 months after the date of the loan. The first payment, $X$, is half as large as the other payments. Interest is calculated at an annual effective rate of 5% for the first 4.5 years and 3% thereafter. Determine $X$.

**A.** 640

**B.** 648

**C.** 656

**D.** 664

**E.** 672

**7**. Chris makes annual deposits into a bank account at the beginning of each year for 20 years. Chris' initial deposit is equal to 100, with each subsequent deposit $k$% greater than the previous year's deposit. The bank credits interest at an annual effective rate of 5%. At the end of 20 years, the accumulated amount in Chris' account is equal to 7276.35. Given $k > 5$, calculate $k$.

**A.** 8.06

**B.** 8.21

**C.** 8.36

**D.** 8.51

**E.** 8.68

**8**. Scott deposits 1 at the beginning of each quarter in year 1, 2 at the beginning of each quarter in year 2, ..., 8 at the beginning of each quarter in year 8. One quarter after the last deposit Scott withdraws the accumulated value of the fund and uses it to buy a perpetuity immediate with level payments of $X$ at the end of each year. All calculations assume a nominal interest rate of 10% per annum compounded quarterly. Calculate $X$.

**A.** 19.4

**B.** 19.9

**C.** 20.4

**D.** 20.9

**E.** 21.4

## Solutions to Practice Examination 14

**1**. The force of interest for Fund X is $\ln(1.05) = 0.04879$ and the force of interest for Fund Y is $.08/(1+.08t)$. The forces of interest are equal when $t = 8$, and at $t = 8$ the difference in the accounts is $10000(1 + 8(.08)) - 10000(1.05)^8 = 1625.44$. **A**.

**2**. Since the two values are equal, $75e^{\int_0^3 \delta_t\,dt} = 150e^{-\int_3^5 \delta_t\,dt}$, so that $75\dfrac{k+6}{k} = 150\dfrac{k+6}{k+10}$ which gives $k = 10$. So $X = 75e^{\int_0^3 \delta_t\,dt} = 120$. **D**.

**3**. From the information, $X = 800(1 + i/4)^{-20}$ and $X = 1120(1 + i/2)^{-20}$. Equating these two values gives $i = 0.069$, and $X = 568.24$. **C**.

**4**. From John's information the inheritance amount $H$ satisfies $H = 2500\ddot{a}_{\overline{10}|.08} = 18117.22$. Thus $18117.22(1.09)^2 = Za_{\overline{15}|.08}$, and $Z = 2514.76$. **C**.

**5**. The equations of value are $100s_{\overline{40}|.08} = Xa_{\overline{15}|.08}$ and $100s_{\overline{40}|.10} = Ya_{\overline{15}|.10}$, from which $X = 3026.54$ and $Y = 5818.93$. So $Y - X = 2792.39$. **A**.

**6**. The equation of value is

$$10000 = X(1.05)^{-1/2} + 2X(1.05)^{-1/2}a_{\overline{4}|.05} + (1.05)^{-4.5}2Xa_{\overline{5}|.03},$$

which gives $X = 655.71$. **C**.

**7**. The given information gives $\sum_{j=0}^{19} 100(1+k)^j(1.05)^{20-j} = 7276.35$, from which $k = 0.0836$. **C**.

**8**. The rate of 10% per annum compounded quarterly is equivalent to 10.38% compounded annually. Thus $\ddot{s}_{\overline{4}|.025}(Is)_{\overline{8}|.1038} = X/.1038$, from which $X = 20.42$. **C**.

## §25. Practice Examination 15

**1**. Jason deposits 3960 into a bank account at $t = 0$. The bank credits interest at the end of each year at a force of interest $\delta_t = \dfrac{1}{8+t}$. Interest can be reinvested at an annual effective rate of 7%. The total accumulated amount at time $t = 3$ is equal to $X$. Calculate $X$.

**A.** 5394

**B.** 5465

**C.** 5551

**D.** 5600

**E.** 5685

**2**. 100 is deposited into an investment account on January 1, 1998. You are given the following information on investment activity that takes place during the year.

| | April 19, 1998 | October 30, 1998 |
|---|---|---|
| Value immediately prior to deposit | 95 | 105 |
| Deposit | $2X$ | $X$ |

The amount in the account on January 1, 1999 is 115. During 1998 the dollar weighted return is 0% and the time weighted return is $y$. Calculate $y$.

**A.** –1.5%

**B.** –0.7%

**C.** 0.0%

**D.** 0.7%

**E.** 1.5%

**3**. Eric deposits 12 into a fund at time 0 and an additional 12 into the same fund at time 10. The fund credits interest at an annual effective rate of $i$. Interest is payable annually and reinvested at an annual effective rate of $0.75i$. At time 20 the accumulated amount of the reinvested interest payments is equal to 64. Calculate $i$, $i > 0$.

**A.** 8.8%

**B.** 9.0%

**C.** 9.2%

**D.** 9.4%

**E.** 9.6%

**4**. A 10 year loan of 10,000 is to be repaid with payments at the end of each year consisting of interest on the loan and a sinking fund deposit. Interest on the loan is charged at a 12% annual effective rate. The sinking fund's annual effective interest rate is 8%. However, beginning in the sixth year the annual effective interest rate on the sinking fund drops to 6%. As a result, the annual payment to the sinking fund is then increased by $X$. Calculate $X$.

**A.** 122

**B.** 132

**C.** 142

**D.** 152

**E.** 162

**5**. Jason and Margaret each take out a 17 year loan of $L$. Jason repays his loan using the amortization method, at an annual effective interest rate of $i$. He makes an annual payment of 500 at the end of each year. Margaret repays her loan using the sinking fund method. She pays interest annually, also at an annual effective interest rate of $i$. In addition, Margaret makes level annual deposits at the end of each year for 17 years into a sinking fund. The annual effective rate on the sinking fund is 4.62% and she pays off the loan after 17 years. Margaret's total payment each year is equal to 10% of the original loan amount. Calculate $L$.

**A.** 4840

**B.** 4940

**C.** 5040

**D.** 5140

**E.** 5240

**6**. Don takes out a 10 year loan of $L$ which he repays with annual payments at the end of each year using the amortization method. Interest on the loan is charged at an annual effective rate of $i$. Don repays the loan with a decreasing series of payments. He repays 1000 in year one, 900 in year two, 800 in year three, . . . , and 100 in year ten. The amount of principal repaid in year three is equal to 600. Calculate $L$.

**A.** 4070

**B.** 4120

**C.** 4170

**D.** 4220

**E.** 4270

**7**. A loan is being amortized by means of level monthly payments at an annual effective interest rate of 8%. The amount of principal repaid in the 12th payment is 1000 and the amount of principal repaid in the $t$th payment is 3700. Calculate $t$.

**A.** 198

**B.** 204

**C.** 210

**D.** 216

**E.** 228

**8**. Laura buys two bonds at time 0. Bond X is a 1000 par value 14 year bond with 10% annual coupons. It is bought at a price to yield an annual effective rate of 8%. Bond Y is a 14 year par value bond with 6.75% annual coupons and a face amount of $F$. Laura pays $P$ for the bond to yield an annual effective rate of 8%. During year 6 the writedown in premium (principal adjustment) on bond X is equal to the writeup in discount (principal adjustment) on bond Y. Calculate $P$.

**A.** 1415

**B.** 1425

**C.** 1435

**D.** 1445

**E.** 1455

## Solutions to Practice Examination 15

**1**. Observe that $e^{\int_0^t \delta_s\,ds} = (8+t)/8$. Thus the growth factors for each of the 3 years are 9/8, 10/9, and 11/10. The total at time $t = 3$ is $3960 + 3960(9/8-1)(1.07)^2 + 3960(10/9-1)(1.07) + 3960(11/10-1) = 5393.53$. **A**.

**2**. The dollar weighted return being 0 means $100 + 2X + X = 115$, so that $X = 5$. The time weighted return is $1 + y = (95/100)(105/105)(115/110) = 0.9931$. So $y = -0.7\%$. **B**.

**3**. Here $12is_{\overline{20}|.75i} + 12is_{\overline{10}|.75i} = 64$. Thus $i = 0.096$. **E**.

**4**. Initially the sinking fund deposit is $10000/s_{\overline{10}|.08} = 690.29$, and after 5 deposits the amount in the sinking fund is $690.29s_{\overline{5}|.08} = 4049.68$. The new sinking fund deposit is therefore $(10000-4049.68(1.06)^5)/s_{\overline{5}|.06} = 812.58$, making $X = 812.58-690.29 = 122.29$. **A**.

**5**. The given information yields $L = 500a_{\overline{17}|i}$ and $0.10L = Li+L/s_{\overline{17}|.0462}$. The second equation gives $i = 0.06$ and the first gives $L = 5238.63$. **E**.

**6**. Here $L = 100(Da)_{\overline{10}|i}$, and the loan balance after the second payment is $100(Da)_{\overline{8}|i}$, so the principal part of the third payment is $600 = 800-100i(Da)_{\overline{8}|i}$. Thus $i = 0.0687$ and $L = 4270.76$. **E**.

**7**. An annual effective rate of 8% is equivalent to 7.72% compounded monthly. Here $pv^{n+1-12} = 1000$ and $pv^{n+1-t} = 3700$. Taking ratios gives $v^{12-t} = 3.7$, from which $t = 216.55$. **D**.

**8**. The price of Bond X is $1000 + (100-80)a_{\overline{14}|.08}$. So the writedown in year 6 is $20(1.08)^{-9} = 10.00$. Similarly, $P = F+(.0675-.08)Fa_{\overline{14}|.08}$, so that $.0125F(1.08)^{-9} = 10$, from which $F = 1599.20$ and $P = 1435.11$. **C**.

## §26. Practice Examination 16

**1**. A 1000 par value 18 year bond with annual coupons is bought to yield an annual effective rate of 5%. The amount for amortization of premium in the 10th year is 20. The book value of the bond at the end of year 10 is $X$. Calculate $X$.

**A.** 1180

**B.** 1200

**C.** 1220

**D.** 1240

**E.** 1260

**2**. Ernie makes deposits of 100 at time 0, and $X$ at time 3. The fund grows at a force of interest $\delta_t = \dfrac{t^2}{100}$, $t > 0$. The amount of interest earned from time 3 to 6 is $X$. Calculate $X$.

**A.** 385

**B.** 485

**C.** 585

**D.** 685

**E.** 785

**3**. Mike buys a perpetuity immediate with varying annual payments. During the first 5 years, the payment is constant and equal to 10. Beginning in year 6 the payments start to increase. For year 6 and all future years, the current year's payment is $K\%$ larger than the previous year's payment. At an annual effective interest rate of 9.2% the perpetuity has a present value of 167.50. Calculate $K$, given $K < 9.2$.

**A.** 4.0

**B.** 4.2

**C.** 4.4

**D.** 4.6

**E.** 4.8

**4**. A 10 year loan of 2000 is to be repaid with payments at the end of each year. It can be repaid under the following two options.

(1) Equal annual payments at an annual effective rate of 8.07%.

(2) Installments of 200 each year plus interest on the unpaid balance at an annual effective rate of $i$.

The sum of the payments under option (1) equals the sum of the payments under option (2). Determine $i$.

**A.** 8.75%

**B.** 9.00%

**C.** 9.25%

**D.** 9.50%

**E.** 9.75%

**5**. A loan is amortized over five years with monthly payments at a nominal interest rate of 9% compounded monthly. The first payment is 1000 and is to be paid one month from the date of the loan. Each succeeding monthly payment will be 2% lower than the prior payment. Calculate the outstanding loan balance immediately after the 40th payment is made.

**A.** 6751

**B.** 6889

**C.** 6941

**D.** 7030

**E.** 7344

**6**. To accumulate 8000 at the end of $3n$ years, deposits of 98 are made at the end of each of the first $n$ years, and 196 at the end of each of the next $2n$ years. The annual effective rate of interest is $i$. You are given $(1 + i)^n = 2.0$. Determine $i$.

**A.** 11.25%

**B.** 11.75%

**C.** 12.25%

**D.** 12.75%

**E.** 13.25%

**7**. Olga buys a 5 year increasing annuity for $X$. Olga will receive 2 at the end of the first month, 4 at the end of the second month, and for each month thereafter the payment increases by 2. The nominal interest rate is 9% convertible quarterly. Calculate $X$.

**A.** 2680

**B.** 2730

**C.** 2780

**D.** 2830

**E.** 2880

**8**. You are given the following information about the activity in two different investment accounts.

Account K

| Date | Fund Value before activity | Activity: Deposit | Activity: Withdrawal |
|---|---|---|---|
| January 1, 1999 | 100 | | |
| July 1, 1999 | 125 | | $X$ |
| October 1, 1999 | 110 | $2X$ | |
| December 31, 1999 | 125 | | |

Account L

| Date | Fund Value before activity | Activity: Deposit | Activity: Withdrawal |
|---|---|---|---|
| January 1, 1999 | 100 | | |
| July 1, 1999 | 125 | | $X$ |
| December 31, 1999 | 105.8 | | |

During 1999 the dollar weighted return for investment account K equals the time weighted return for investment account L, which equals $i$. Calculate $i$.

**A.** 10%

**B.** 12%

**C.** 15%

**D.** 18%

**E.** 20%

## Solutions to Practice Examination 16

**1**. If the coupon rate is $p$ then $p > 5\%$ and $(1000p - 50)(1.05)^{-9} = 20$ so that $p = 0.081$. Thus $X = 1000 + 31a_{\overline{8}|.05} = 1200.36$. **B**.

**2**. Note that $e^{\int_0^t \delta_s\, ds} = e^{t^3/300}$. The amount in the account at time 3 is $100e^{27/300} + X$ while the amount in the account at time 6 is $100e^{216/300} + Xe^{216/300-27/300}$. Thus $X$ is the difference of these 2 amounts, which gives $X = 784.59$. **E**.

**3**. Here $167.50 = 10a_{\overline{5}|} + 10v^5 \sum_{j=1}^{\infty}(1+K)^j v^j$, so that $K = 0.040$. **A**.

**4**. Under option (1) the total of all payments is $10(2000/a_{\overline{10}|.0807}) = 2990$. Under option (2) the total of all payments is $200(10) + i\sum_{j=0}^{9}(2000 - 200j)$. Equating these two gives $i = 0.090$. **B**.

**5**. The $k$th payment is $1000(.98)^{k-1}$. The present value at the time of the 40th payment of the remaining payments is $\sum_{j=1}^{20} 1000(.98)^{40+j-1}v^j = 6889.11$. **B**.

**6**. The equation of value is $98s_{\overline{n}|}v^{-2n} + 196s_{\overline{2n}|} = 8000$ Using $s_{\overline{n}|} = (v^{-n} - 1)/i = 1/i$ and $s_{\overline{2n}|} = 3/i$ gives $i = 0.1225$. **C**.

**7**. An interest rate of 9% convertible quarterly is equivalent to 8.93% compounded monthly. Thus $X = 2(Ia)_{\overline{60}|.0893/12} = 2729.50$. **B**.

**8**. The dollar weighted return for account K is $100(1+i) - X(1+i/2) + 2X(1+i/4) = 125$, or $i = (25 - X)/100$. The time weighted return for account L is $1 + i = (125/100)(105.8/(125 - X))$. Substituting the expression for $i$ into the second equation gives $X = 10$ and $i = 0.15$. **C**.

# §27. Practice Examination 17

**1**. David can receive one of the following two payment streams.

(1) 100 at time 0, 200 at time $n$, and 300 at time $2n$

(2) 600 at time 10.

At an annual effective interest rate of $i$, the present values of the two streams are equal. Given $v^n = 0.75941$, determine $i$.

**A.** 3.5%

**B.** 4.0%

**C.** 4.5%

**D.** 5.0%

**E.** 5.5%

**2**. A man turns 40 today and wishes to provide supplemental retirement income of 3000 at the beginning of each month starting on his 65th birthday. Starting today, he makes monthly contributions of $X$ to a fund for 25 years. The fund earns a nominal rate of 8% compounded monthly. Each 1000 will provide for 9.65 of income at the beginning of each month starting on his 65th birthday until the end of his life. Calculate $X$.

**A.** 324.73

**B.** 326.89

**C.** 328.12

**D.** 355.45

**E.** 450.65

**3**. Payments are made to an account at a continuous rate of $(8k+tk)$, where $0 \le t \le 10$. Interest is credited at a force of interest $\delta_t = \frac{1}{8+t}$. After 10 years the account is worth 20,000. Calculate $k$.

**A.** 111

**B.** 116

**C.** 121

**D.** 126

**E.** 131

**4**. You have decided to invest in two bonds. Bond X is an $n$-year bond with semi-annual coupons, while bond Y is an accumulation bond redeemable in $n/2$ years. The desired yield rate is the same for both bonds. You also have the following information. Bond X has a par value of 1000, the present value of the redemption value is 381.50, and the ratio of the semi-annual bond rate to the desired semi-annual yield rate, $r/i$, is 1.03125. Bond Y has the same redemption value as bond X, and the price to yield is 647.80. What is the price of bond X?

**A.** 1019

**B.** 1029

**C.** 1050

**D.** 1055

**E.** 1072

**5**. At time $t = 0$, Sebastian invests 2000 in a fund earning 8% convertible quarterly, but payable annually. He reinvests each interest payment in individual separate funds each earning 9% convertible quarterly, but payable annually. The interest payments from the separate funds are accumulated in a side fund that guarantees an annual effective rate of 7%. Determine the total value of all funds at $t = 10$.

**A.** 3649

**B.** 3964

**C.** 4339

**D.** 4395

**E.** 4485

**6**. Bruce deposits 100 into a bank account. His account is credited interest at a nominal rate of interest $i$ convertible semiannually. At the same time, Peter deposits 100 into a separate account. Peter's account is credited interest at a force of interest $\delta$. After 7.25 years the value of each account is 200. Calculate $i - \delta$.

**A.** 0.12%

**B.** 0.23%

**C.** 0.31%

**D.** 0.39%

**E.** 0.47%

**7**. Kathryn deposits 100 into an account at the beginning of each 4 year period for 40 years. The account credits interest at an annual effective interest rate of $i$. The accumulated amount in the account at the end of 40 years is $X$, which is 5 times the accumulated amount in the account at the end of 20 years. Calculate $X$.

**A.** 4695

**B.** 5070

**C.** 5445

**D.** 5820

**E.** 6195

**8**. Eric deposits $X$ into a savings account at time 0, which pays interest at a nominal rate of $i$ compounded semiannually. Mike deposits $2X$ into a different savings account at time 0 which pays simple interest at an annual rate of $i$. Eric and Mike earn the same amount of interest during the last 6 months of the 8th year. Calculate $i$.

**A.** 9.06%

**B.** 9.26%

**C.** 9.46%

**D.** 9.66%

**E.** 9.86%

## Solutions to Practice Examination 17

**1**. Equating the present values gives $600v^{10} = 100 + 200v^n + 300v^{2n}$, from which $v^{10} = 0.708$ and $i = 0.0351$. **A**.

**2**. He needs $1000(3000/9.65) = 310,880$ at retirement, so $310880 = X\ddot{s}_{\overline{300}|.08/12}$, from which $X = 324.73$. **A**.

**3**. Note that $e^{\int_0^t \delta_s\,ds} = (8+t)/8$. The value today of the $(8k+tk)dt$ deposited at time $t$ is thus $k(8+t)\times 8/(8+t) = 8k$, so the present value of all the deposits is $80k$. The present value of the 20,000 is $20000(8/18) = 8888.88$. Thus $k = 8888.88/80 = 111.11$. **A**.

**4**. Since the bond rate is higher than the yield rate, bond X sells at a premium. Also, at the yield rate, $647.80 = Rv^n$ and $381.50 = Rv^{2n}$, from the information about the present value of the redemption value $R$. Thus $v^n = 381.50/647.80 = 0.5889$. Now the price of bond X is $1000ra_{\overline{2n}|} + 381.50 = (1000r/i)ia_{\overline{2n}|} + 381.50 = 1031.25(1 - v^{2n}) + 381.50 = 1055.10$. **D**.

**5**. The first fund generates $2000(1.02)^4 - 2000 = 164.86$ in interest at the end of each year. The 9% accounts pay interest at an annual rate of 9.31%, thus spinning off $164.86(.0931) = 15.35$ in interest each year, starting in the second year. The total in the accounts is therefore $2000 + 10(164.86) + 15.35(Is)_{\overline{9}|.07} = 4485.49$. **E**.

**6**. The equations of value are $100(1 + i/2)^{14.5} = 200$ for Bruce's account and $100e^{7.25\delta} = 200$ for Peter's. Thus $i = 0.0979$ and $\delta = 0.0956$ so that $i - \delta = 0.0023$. **B**.

**7**. At the end of 20 years (just before the sixth deposit) the account contains $100(v^{-20} + v^{-16} + v^{-12} + v^{-8} + v^{-4}) = 100(v^{-20} - 1)/(1 - v^4)$, while at the end of 40 years, the account contains $100(v^{-40} - 1)/(1 - v^4)$. Since this is five times the previous amount $500(v^{-20} - 1) = 100(v^{-40} - 1)$, from which $v^{-20} = 4$ and $X = 6194.72$. **E**.

**8**. Eric earns $X((1 + i/2)^{16} - (1 + i/2)^{15})$ of interest while Mike earns $2X(i/2) = Xi$. Equating these two gives $(1 + i/2)^{15} = 2$ from which $i = 0.09458$. **C**.

## §28. Practice Examination 18

**1**. John borrows 1000 for 10 years at an annual effective interest rate of 10%. He can repay this loan using the amortization method with payments of $P$ at the end of each year. Instead John repays the 1000 using a sinking fund that pays an annual effective rate of 14%. The deposits to the sinking fund are equal to $P$ minus the interest on the loan and are made at the end of each year for 10 years. Determine the balance in the sinking fund immediately after repayment of the loan.

**A.** 213

**B.** 218

**C.** 223

**D.** 230

**E.** 237

**2**. An association had a fund balance of 75 on January 1 and 60 on December 31. At the end of every month during the year the association deposited 10 from membership fees. There were withdrawals of 5 on February 28, 25 on June 30, 80 on October 15, and 35 on October 31. Calculate the dollar weighted rate of return for the year.

**A.** 9.0%

**B.** 9.5%

**C.** 10.0%

**D.** 10.5%

**E.** 11.0%

**3**. A perpetuity costs 77.1 and makes annual payments at the end of the year. The perpetuity pays 1 at the end of year 2, 2 at the end of year 3, . . . , $n$ at the end of year $(n + 1)$. After year $(n + 1)$ the payments remain constant at $n$. The annual effective interest rate is 10.5%. Calculate $n$.

**A.** 17

**B.** 18

**C.** 19

**D.** 20

**E.** 21

**4**. 1000 is deposited into Fund X, which earns an annual effective rate of 6%. At the end of each year, the interest earned plus an additional 100 is withdrawn from the fund. At the end of the tenth year the fund is depleted. The annual withdrawals of interest and principal are deposited into Fund Y, which earns an annual effective rate of 9%. Determine the accumulated value of Fund Y at the end of year 10.

**A.** 1519

**B.** 1819

**C.** 2085

**D.** 2273

**E.** 2431

**5.** You are given the following table of interest rates.

| Calendar Year of Original Investment | Investment Year Rates (in %) | | | | | Portfolio Rates (in%) |
|---|---|---|---|---|---|---|
| $y$ | $i_1^y$ | $i_2^y$ | $i_3^y$ | $i_4^y$ | $i_5^y$ | $i^{y+5}$ |
| 1992 | 8.25 | 8.25 | 8.4 | 8.5 | 8.5 | 8.35 |
| 1993 | 8.5 | 8.7 | 8.75 | 8.9 | 9.0 | 8.6 |
| 1994 | 9.0 | 9.0 | 9.1 | 9.1 | 9.2 | 8.85 |
| 1995 | 9.0 | 9.1 | 9.2 | 9.3 | 9.4 | 9.1 |
| 1996 | 9.25 | 9.35 | 9.5 | 9.55 | 9.6 | 9.35 |
| 1997 | 9.5 | 9.5 | 9.6 | 9.7 | 9.7 | |
| 1998 | 10.0 | 10.0 | 9.9 | 9.8 | | |
| 1999 | 10.0 | 9.8 | 9.7 | | | |
| 2000 | 9.5 | 9.5 | | | | |
| 2001 | 9.0 | | | | | |

A person deposits 1000 on January 1, 1997. Let the following be the accumulated value of 1000 on January 1, 2000

(1) $P$: under the investment year method

(2) $Q$: under the portfolio yield method

(3) $R$: where the balance is withdrawn at the end of every year and is reinvested at the new money rate

Determine the ranking of P, Q, and R.

**A.** $P > Q > R$

**B.** $P > R > Q$

**C.** $Q > P > R$

**D.** $R > P > Q$

**E.** $R > Q > P$

**6**. At an annual effective interest rate of $i$, $i > 0$, both of the following annuities have a present value of $X$.

(1) a 20 year annuity immediate with annual payments of 55

(2) a 30 year annuity immediate with annual payments that pays 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years

Calculate $X$.

**A.** 575

**B.** 585

**C.** 595

**D.** 605

**E.** 615

**7**. Eric and Jason each sell a different stock short at the beginning of the year for a price of 800. The margin requirement for each investor is 50% and each will earn an annual effective interest rate of 8% on his margin account. Each stock pays a dividend of 16 at the end of the year. Immediately thereafter, Eric buys back his stock at a price of $(800 - 2X)$ and Jason buys back his stock at a price of $(800 + X)$. Eric's annual effective yield $i$ on the short sale is twice Jason's annual effective yield. Calculate $i$.

**A.** 4%

**B.** 6%

**C.** 8%

**D.** 10%

**E.** 12%

**8**. A 30 year loan of 1000 is repaid with payments at the end of each year. Each of the first ten payments equals the amount of interest due. Each of the next ten payments equals 150% of the amount of interest due. Each of the last ten payments is $X$. The lender charges interest at an annual effective rate of 10%. Calculate $X$.

**A.** 32

**B.** 57

**C.** 70

**D.** 97

**E.** 117

## Solutions to Practice Examination 18

**1**. Here $Pa_{\overline{10}|.10} = 1000$ so that $P = 162.74$, and $(P - 100)s_{\overline{10}|.14} = 1213.22$ is the total amount in the sinking fund. Thus 213.22 remains. **A**.

**2**. The dollar weighted return satisfies $75(1+i)+\sum_{j=0}^{11} 10(1+ij/12)-5(1+10i/12)-25(1+6i/12)-80(1+2.5i/12)-35(1+2i/12) = 60$ so that $(75+55-50/12-150/12-200/12-70/12)i = 10$, and $i = 0.11$. **E**.

**3**. Here $77.1 = v((Ia)_{\overline{n}|} + v^n n/i) = a_{\overline{n}|}/i$. Thus $n = 19$. **C**.

**4**. The deposit into Fund Y at the end of year $k$ is $100 + (60 - 6(k - 1))$. Thus the accumulation in Fund Y is $166s_{\overline{10}|} - 6(Is)_{\overline{10}|} = 2084.67$. **C**.

**5**. The yearly interest rates under $P$ are 9.5, 9.5, and 9.6; under $Q$ are 8.35, 8.6, and 8.85; under $R$ are 9.5, 10.0, and 10.0. Thus $R > P > Q$. **D**.

**6**. From the given information, $X = 55a_{\overline{20}|} = 55(a_{\overline{10}|} + v^{10}a_{\overline{10}|})$ and $X = 30a_{\overline{10}|} + v^{10}60a_{\overline{10}|} + 90v^{20}a_{\overline{10}|}$. Equating these two gives $55(1+v^{10}) = 30+60v^{10}+90v^{20}$, from which $v^{10} = 0.5$, $i = 0.07177$, and $X = 574.74$. **A**.

**7**. Eric's return is $(32-16+800-(800-2X))/400$ and Jason's is $(32-16+800-(800+X))/400$. Thus $2X + 16 = 2(16 - X)$, from which $X = 4$ and Eric's return is $24/400 = 0.06$. **B**.

**8**. After the first 10 payments the borrower still owes 1000. The middle scheme reduces the loan balance by a factor of $1 - i/2 = .95$ with each payment, so the remaining balance to be paid by the final ten payments is $1000(.95)^{10} = 598.74$. Thus $X = 598.74/a_{\overline{10}|} = 97.44$. **D**.