Lecture Notes
on
Ordinary Differential Equations

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§0. Introduction

These notes provide an introduction to both the quantitative and qualitative methods of solving ordinary differential equations. Emphasis is placed on first and second order equations with constant coefficients. The equations studied are often derived directly from physical considerations in applied problems.

This is not designed as a mathematical theory course, but rather as a workbook in the application of these particular techniques. More advanced mathematical treatises can be consulted for a theoretical treatment of the subject matter.

This is also not a course in using computers to solve differential equations. Students are encouraged to use such devices, as appropriate, in working on the problems here. The emphasis here is on the conceptual reasons that the methods used by computers work. The specific objectives are as follows.

(1) To be able to identify and classify an ordinary differential equation.

(2) To understand what it means for a function to be a solution of an ordinary differential equation.

(3) To be able to find the solution to certain simple ordinary differential equations.

(4) To be able to discover some properties of the solution of an ordinary differential equation without actually finding the solution.

(5) To be able to derive an ordinary differential equation as the mathematical model for a physical phenomenon.

The reader should take the time to work through all of the exercises and problems in order to understand the steps thoroughly. Doing so will virtually guarantee success in learning the material.

§1. A Simple Growth Model

A simple model for growth is introduced and the corresponding differential equation is derived and solved.

Example 1–1. The number $P(t)$ of bacteria in a colony at time t is to be studied. The culture contains an ample food supply and there are no predators for the bacteria in the culture. The time frame under consideration is assumed to be short relative to the life expectancy of an individual bacterium. Under these conditions the only way in which the population size changes is by birth of new bacteria. One reasonable assumption is that the number of births is proportional to the number of bacteria currently alive. By equating rates of change, this assumption leads to the model

$$\frac{d}{dt}P(t) = kP(t)$$

for all $t > 0$ where k is the proportionality constant. The constant k has the interpretation as the birth rate for the colony.

The equation derived above is called an ordinary differential equation because the equation expresses a relationship between the unknown function $P(t)$ and its derivative. (The adjective ‘ordinary’ means that only ordinary derivatives appear in the equation; there are no partial derivatives appearing in the equation.) More completely, an **ordinary differential equation** is an equation which expresses a relationship between the derivatives of an unknown function, the independent variable, and the unknown function itself.

The equation above is a **first order** differential equation because only the first derivative of the unknown function $P(t)$ appears in the equation. The equation has **constant coefficients** because the unknown function $P(t)$ and its derivatives are multiplied only by constants, not by functions of the independent variable t. The equation is **linear** because the unknown function $P(t)$ and its derivatives appear only to the first power. The equation is also **homogeneous** because the function $P(t)$ which is identically 0 for all t satifies the equation.

Exercise 1–1. Verify that if $P(t) = 0$ for all t then $\frac{d}{dt}P(t) = kP(t)$ for all $t > 0$. Thus the differential equation $\frac{d}{dt}P(t) = kP(t)$ is homogeneous.

A **solution** of an ordinary differential equation is a function which satisfies the equation at all points in the domain of the function.

Exercise 1–2. Show that the function $P(t) = e^{kt}$ solves the differential equation above. (Note that the domain of the function e^{kt} is all real numbers t.) Are there any other solutions?

The previous exercise points to a two step method for solving homogeneous first order linear ordinary differential equations with constant coefficients.

(1) Try a solution of the form e^{mt} and find the value of m that makes this function a solution.

(2) The general solution is then of the form Ce^{mt} where C is an arbitrary constant.

This procedure shows that, with one exception, the solution of a homogeneous first order linear ordinary differential equation with constant coefficients is an exponential function.

Exercise 1–3. What is the one exception?

Example 1–2. In order to solve the equation $\frac{d}{dt}A(t) = 5A(t)$ try a solution of the form e^{mt}. Substituting this trial solution into the equation leads to

$$me^{mt} = 5e^{mt}$$

and this equation holds for all t only if $m = 5$. The general solution of the equation is $A(t) = Ce^{5t}$.

Example 1–3. In the bacteria problem suppose the initial population is 100 and the growth rate is 5%. Then $P(t) = Ce^{0.05t}$ is the general solution to the differential equation. Since the initial population is 100, $P(0) = 100$. Using this fact and the general solution shows that $C = 100$. Hence $P(t) = 100e^{0.05t}$.

In the case of the previous example the given information $P(0) = 100$ is called an **initial condition**, for obvious reasons. The initial condition is used to find the appropriate value of the arbitrary constant that appears in the general solution of the differential equation.

Problems

Problem 1–1. Solve the equation $\frac{d}{dt}B(t) = 3B(t)$ for $t > 0$ under the assumption that $B(0) = 50$. What is the domain of the solution?

Problem 1–2. Solve the equation $\frac{d}{dt}B(t) = 3B(t)$ for $t > 0$ under the assumption that $B'(0) = 50$. What is the domain of the solution?

Problem 1–3. True or False: The function $x(t) = e^{-t} + 1$ is a solution of the differential equation $\frac{d^2}{dt^2}x(t) + x(t) = 1$.

Problem 1–4. For what value(s) of λ is $\cos \lambda t$ a solution of the equation $\frac{d^2}{dt^2}f(t) + 9f(t) = 0$?

Problem 1–5. True or False: The function $x(t) = t$ is a solution of the equation $x''(t) - tx'(t) + x(t) = 0$.

Problem 1–6. True or False: The equation $x''(t) + 3x(t) = 5$ is a homogeneous ordinary differential equation.

Problem 1–7. For what value(s) of m is e^{mt} a solution of the equation $x''(t) + 3x'(t) + 2x(t) = 0$?

Problem 1–8. Let $A(t)$ denote the amount of money in an interest earning bank account at time t. Then $\frac{d}{dt}A(t) = kA(t)$ is a reasonable model for $A(t)$. Find k if the account pays interest at a rate of 5% compounded daily. (In this setting the constant k is called the **force of interest**.) If the initial amount in the account is \$1000, how much is in the account after 10 years?

Problem 1–9. Let $A(t)$ denote the amount of a radioactive substance that is present in a sample at time t. The usual model for radioactive decay is that $\frac{d}{dt}A(t) = kA(t)$. The rate of decay for radioactive isotopes is usually specified in terms of the half-life. The half-life of an isotope is the time required for one-half of the initial amount to decay. If the half life of an isotope is 12 years, what is k? What is the general relationship between the half-life of the isotope and k?

Problem 1–10. Suppose the food supply for a bacteria colony is limited and can only support 2000 bacteria. Write a differential equation that could be a model for this situation. Is your equation first order? Linear?

Problem 1–11. Solve the equation $\frac{d}{dt}A(t) = 6t + 7$ with the initial condition $A(0) = 3$. Hint: Simply integrate both sides.

Solutions to Problems

Problem 1–1. The general solution is of the form $B(t) = Ce^{3t}$. Since $B(0) = 50$, $50 = Ce^{3\cdot 0}$, and so $C = 50$. The solution is $B(t) = 50e^{3t}$. The domain is the set of all real numbers.

Problem 1–2. The general solution is $B(t) = Ce^{3t}$. Since $B'(0) = 50$, $50 = 3Ce^{3\cdot 0}$ so $C = 50/3$. The solution is $B(t) = (50/3)e^{3t}$. The domain is the set of all real numbers.

Problem 1–3. True, by substitution.

Problem 1–4. By substitution, $\cos \lambda t$ is a solution if and only if $\lambda^2 + 9 = 0$.

Problem 1–5. Since $\dfrac{d^2}{dt^2}tt = 0$ and $\dfrac{d}{dt}tt = 1$, substitution shows that the answer is true.

Problem 1–6. The zero function is not a solution, so the equation is not homogeneous. Hence the answer is false.

Problem 1–7. Since $\dfrac{d^2}{dt^2}e^{mt} = m^2e^{mt}$ and $\dfrac{d}{dt}e^{mt} = me^{mt}$ the requirement on m becomes $m^2 + 3m + 2 = 0$, which is satisfied for $m = -2$ and $m = -1$.

Problem 1–8. The general solution is $A(t) = Ce^{kt}$. Assume time is measured in days. Then $A(1) = (1 + 0.05/365)A(1)$ by the definition of daily compounding of interest. Substituting the general solution gives $e^k = 1 + 0.05/365$ so $k = \ln(1 + 0.05/365)$. If $A(0) = 1000$ then $C = 1000$ too and after 10 years (or 3650 days) the amount in the account is $1000e^{3650\ln(1+0.05/365)} = 1648.66$.

Problem 1–9. The general solution is $A(t) = Ce^{kt}$. Note that $C = A(0)$. If the half life is 12 years then $A(12) = A(0)/2$. Using the general form of the solution gives $A(0)/2 = A(0)e^{12k}$ from which $k = (1/12)\ln(1/2)$. More generally the same argument shows that $k = (1/T)\ln(1/2)$, where T is the half-life.

Problem 1–10. Considering the way in which $\frac{d}{dt}P(t)$ should depend on $P(t)$ shows that $\frac{d}{dt}P(t)$ should be 0 if $P(t) = 0$ or $P(t) = 2000$, while this derivative should be positive for $0 < P(t) < 2000$ and negative otherwise. One model satisfying these conditions is $\frac{d}{dt}P(t) = kP(t)(2000 - P(t))$. This equation is first order but not linear. There are many others.

Problem 1–11. Integrating both sides of the original equation from 0 to t gives $A(t) - A(0) = 3t^2 + 7t$. Using $A(0) = 3$ gives $A(t) = 3t^2 + 7t + 3$.

Solutions to Exercises

Exercise 1–1. Just substitute the zero function on both sides of the equation and verify that equality holds.

Exercise 1–2. Here just substitute again and check for equality. Since $\frac{d}{dt}e^{kt} = ke^{kt}$, the function e^{kt} is a solution. If $g(t)$ is any other solution, what is $\frac{d}{dt}g(t)e^{-kt}$?

Exercise 1–3. The function which is identically zero is also a solution.

§2. First Order Autonomous Equations

The separation of variables technique is introduced in order to solve first order autonomous equations.

Example 2–1. A marble is dropped from a tower. The two forces acting on the marble are gravity and air resistance. As a simple model for air resistance, the force due to air resistance is assumed to be proportional to velocity. Assume that the upward direction is the positive direction. If the mass of the marble is m, Newton's Law then gives the equation

$$m\frac{d}{dt}v(t) = -kv(t) - mg$$

for the velocity $v(t)$ of the marble at time $t > 0$. Here g is the gravitational acceleration constant and $k > 0$ is the proportionality constant which determines the magnitude of the air resistance effect.

Exercise 2–1. What would be an appropriate initial condition here?

The equation for $v(t)$ is a linear first order equation with constant coefficients. This equation is not homogeneous. The equation is **autonomous** because the independent variable t appears only through the function $v(t)$ and its derivatives.

First order autonomous equations can always be solved (in principle) by the **separation of variables** technique.

(1) Isolate the derivative of the unknown function on one side of the equation.

(2) Divide by the quantity on the other side of the equation.

(3) Integrate both sides of the resulting equation making use of the Fundamental Theorem of Calculus and the initial condition.

Example 2–2. To illustrate the method the equation $\frac{d}{dt}v(t) = v(t) - 1$ with initial condition $v(0) = 0$ will be solved. Dividing both sides by $v(t) - 1$ gives

$$\frac{\frac{d}{dt}v(t)}{v(t) - 1} = 1$$

and both sides can now be integrated with respect to t. Since the initial condition is specified at time $t = 0$, the integration is from $t = 0$ to an arbitrary point $t = s$. This gives

$$\int_0^s \frac{v'(t)}{v(t) - 1}\,dt = \int_0^s 1\,dt.$$

After integration, using the fact that $v(0) = 0$, this becomes $\ln|v(s) - 1| = s$ or $|v(s) - 1| = e^s$. Since $v(0) = 0$, $v(s) - 1$ is at least initially negative, so removing the absolute values finally gives $v(s) = 1 - e^s$ for $s \geq 0$ as the solution.

Exercise 2–2. Verify that this is indeed a solution of the differential equation that satisfies the initial condition.

The success of the separation of variables technique depends on the ability to integrate the resulting expression. This can at times be difficult or even impossible.

First order homogeneous linear equations with constant coefficients are also autonomous. The separation of variables technique provides an alternate means of solving such equations.

Example 2–3. Computer software programs, such as Maple and Mathematica, are capable solving many of the simple differential equations considered here. The following are examples of Maple commands that would be used. Note that all Maple commands must end with a semicolon. The equation $x'(t) = 2x(t)$ with initial condition $x(0) = 5$ would be solved with

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dsolve({diff(x(t),t) = 2 * x(t), x(0) = 5}, x(t));,
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while the same equation with initial condition $x'(0) = 5$ would be solved with the command

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dsolve({diff(x(t),t) = 2 * x(t), D(x)(0) = 5}, x(t));.
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The command

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dsolve({diff(v(t),t) = -k * v(t) - m * g, v(0) = 0}, v(t));
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solves the equation of the first example of this section.

Problems

Problem 2–1. Solve the equation $\frac{d}{dt}A(t) = A(t)^2 - 1$ with $A(0) = 0$. Hint: Use partial fractions.

Problem 2–2. Find all solutions of the equation $\frac{d}{dt}A(t) = A(t)^2 - 1$. What are the constant solutions?

Problem 2–3. Find the solution of the equation $A'(t) + A(t) - 1 = 0$ which satisfies the condition $A(0) = 0$.

Problem 2–4. Solve the equation $m\frac{d}{dt}v(t) = -kv(t) - mg$ with initial condition $v(0) = 0$ when $k = 0.1$ and $m = 1$. (Recall that $g = 9.8$ meters per second per second.) Plot your solution. Compare your solution to the case in which there is no air resistance ($k = 0$). What is the difference?

Problem 2–5. Let $P(t)$ be the population size for a bacteria colony at time t. The **logistic model** is that $\frac{d}{dt}P(t) = kP(t)(M - P(t))$, where $k > 0$ and $M > 0$ are constants. Solve this equation when $k = 1$ and $M = 1000$ with $P(0) = 100$. Graph your solution. Hint: Use partial fractions.

Problem 2–6. Solve the equation $\frac{d}{dt}A(t) = A(t) + 1$ with the initial condition $A(0) = 5$.

Problem 2–7. Solve the equation $\frac{d}{dt}A(t) = (A(t))^2 + 1$ with the initial condition $A(0) = 1$. Hint: $\int \frac{dx}{1 + x^2} = \arctan(x)$. Sketch a graph of the solution $A(t)$. For which values of t do you think the differential equation is valid?

Problem 2–8. Solve the equation $\frac{d}{dt}x(t) = 5x(t)$ with $x(0) = 10$ using the separation of variables technique.

Problem 2–9. A bank account for a large corporation accrues interest with a force of interest of 5%. The average inflow is 5000 per unit time. If the initial account balance is 1,000,000, find the balance at time 20. Hint: Why is $\frac{d}{dt}A(t) = kA(t) + D$ a suitable differential equation for the amount $A(t)$ in the account at time t?

Problem 2–10. You have a balance due of \$3000 on your credit card. If the interest rate is 18% per annum compounded monthly and you pay \$150 per month, when will your credit card debt be paid in full? Assume you make no additional charges on the account.

Problem 2–11. Newton's Law of Cooling states that the rate of change of temperature of a body submersed in a bath is proportional to the difference of temperature between the body and the bath. If a kettle of 100 degree water cools to a temperature of 80 degrees in 30 minutes when placed in a room of constant temperature 25, what is the proportionality constant? How long will it take for the kettle to reach room temperature?

Problem 2–12. Solve the equation $\frac{d}{dt}B(t) = tB(t)$ with $B(0) = 1$. Note that although this is not an autonomous equation, the separation of variables technique still works if both sides are divided by $B(t)$. Equations such as this one are called **separable**.

Problem 2–13. An injection of medicine is immediately absorbed into the patients bloodstream. The medicine is removed from the blood stream by the patients metabolism. The removal rate is proportional to the amount of medicine in the bloodstream. Suppose the amount of the initial injection is d and the proportionality constant for the removal rate is $r > 0$. If $M(t)$ is the amount of medicine in the patients bloodstream t minutes after the injection, find a differential equation for $M(t)$ and solve it. Find a general formula for $M(t)$ if the patient receives an injection of size d every 120 minutes.

Solutions to Problems

Problem 2–1. After separating variables make the substitution $x = A(t)$ in the integral. The integral to be computed is then $\int_0^{A(t)} \frac{dx}{x^2-1}$. Using partial fractions gives $\frac{1}{x^2-1} = \frac{-1/2}{x+1} + \frac{1/2}{x-1}$. Using this to compute the integral, and then resubstituting $x = A(t)$ gives $(1/2)\ln|(A(t)-1)/(A(t)+1)| = t$. The initial condition $A(0) = 0$ shows that the quantity inside the absolute value is (initially) negative. Solving gives $A(t) = (1-e^{2t})/(1+e^{2t})$.

Problem 2–2. The constant solutions are $A(t) = 1$ and $A(t) = -1$. Using the method of the previous problem shows that generally $(1/2)\ln|(A(t)-1)/(A(t)+1)| = t + A(0)$, from which $|(A(t)-1)/(A(t)+1)| = e^{2t+A(0)}$. How can the absolute value signs be removed? Is there a choice of $A(0)$ which gives the constant solutions?

Problem 2–3. By separation of variables, $A'(t)/(1-A(t)) = 1$, so that after integration, $-\ln|1-A(t)| = t$. Now $A(0) = 0$, and since $1 - A(0) = 1 > 0$, the absolute value signs can be removed to give $A(t) = 1 - e^{-t}$.

Problem 2–4. Separating variables and integrating gives $(-1/k)\ln|(-kv(t) - mg)/(-mg)| = t$. Since $v(0) = 0$ the quantity inside the absolute values is positive. Hence, $v(t) = (mg/k)(e^{-kt} - 1)$. When $k = 0$ the solution is more easily found by a single integration: $v(t) = -mgt$. When $k \neq 0$, the speed never exceeds mg/k, while when $k = 0$ the speed becomes arbitrarily large.

Problem 2–5. The partial fractions expansion is $\frac{1}{P(t)(M-P(t))} = \frac{1/M}{P(t)} + \frac{1/M}{M-P(t)}$. Using this gives $(1/M)\ln|(P(t)/(M-P(t))/(P(0)/(M-P(0)))| = kt$. In the particular case given, $P(t) = \frac{1000}{1+9e^{-1000kt}}$.

Problem 2–6. This is an autonomous equation. Separating variables and integrating gives $\ln|(A(t)+1)/(A(0)+1)| = t$. Since $A(0) = 5$, $A(t)+1$ is positive (at least initially), so exponentiating both sides, removing absolute values, and solving gives $A(t) = (A(0)+1)e^t - 1$. Using the initial condition $A(0) = 5$ gives finally $A(t) = 6e^t - 1$.

Problem 2–7. This is an autonomous equation. Separating variables and integrating gives $\arctan(A(t)) - \arctan(A(0)) = t$. Since $A(0) = 1$, and $\arctan(1) = \pi/4$, $A(t) = \tan(t+\pi/4)$. The graph of the tangent function has vertical asymptotes at odd multiples of $\pi/2$. The differential equation will fail to be satisfied at the first vertical asymptote that is reached. The equation can only hold for $0 \le t < \pi/4$. After that, there will be other solutions not necessarily related to the initial one.

Problem 2–8. The solution is $x(t) = 10e^{5t}$, as can be found by the earlier technique also.

Problem 2–9. In the hint, $k = 0.05$ represents the force of interest and $D = 5000$ is the rate at which deposits are made. The solution is $A(t) = 1,100,000e^{0.05t} -$

100, 000. At time $t = 20$ this gives 2, 890, 110.01.

Problem 2–10. A differential equation for the balance $B(t)$ owing on the card at time t is $\frac{d}{dt}B(t) = 0.015B(t) - 150$. Here time is measured in months and the force of interest is approximately $0.18/12 = 0.015$. The payments are assumed to be made continuously. Solving, using $B(0) = 3000$, gives $B(t) = 10,000 - 7,000e^{0.015t}$. The time t at which the balance is 0 is then 23.77 months, say 24 months.

Problem 2–11. The differential equation for the temperature $K(t)$ of the kettle at time t is $\frac{d}{dt}K(t) = m(K(t) - 25)$, and the information given is that $K(0) = 100$ and $K(30) = 80$ (time is measured in minutes). Solving the differential equation gives $\ln|(K(t) - 25)/(K(0) - 25)| = mt$. The information supplied shows that initially the quantity in the absolute values is positive. Using $K(0) = 100$ gives $K(t) = 25 + 75e^{mt}$. Using $K(30) = 80$ gives $80 = 25 + 75e^{30m}$ from which $m = (1/30)\ln(55/75)$. So finally $K(t) = 25 + 75e^{(t/30)\ln(55/75)}$. Since the exponential is always positive, the kettle never reaches room temperature.

Problem 2–12. Separating variables and integrating gives $\ln|B(t)| = t^2/2$. Using the initial condition shows that $B(t)$ is (initially) positive. Solving gives $B(t) = e^{t^2/2}$.

Problem 2–13. Here $\frac{d}{dt}M(t) = -rM(t)$ with $M(0) = d$. The differential equation is the same in each successive dosing period, only the initial condition changes.

Solutions to Exercises

Exercise 2–1. Since the marble is dropped, $v(0) = 0$. Here time $t = 0$ is taken to be the time at which the marble is dropped.

Exercise 2–2. Substitution shows that this function is a solution. Since $e^0 = 1$, the initial condition is satisfied too.

§3. Two Approximate Solution Methods

Performing the integration required to solve even simple first order autonomous equations can be very difficult, and occasionally impossible. The Fundamental Theorem of Calculus, when coupled with the differential equation itself, suggests one simple scheme for computing the value of the solution of a differential equation numerically. The underlying idea is for this first computational method is that the graph of a function can be obtained by simply plotting points. A second computational method is based on the idea of using the differential equation to compute the Maclaurin (or Taylor) series for the solution.

To uncover the first computational method, recall that the Fundamental Theorem of Calculus states that for any function $f(x)$ and for any numbers $a < b$, $f(b) = f(a) + \int_a^b f'(x)\,dx$. If the number b is not much larger than a, the value of the integral can be approximated as $f'(a)(b-a)$. This is because for b near a the values of $f'(x)$ are all about equal to $f'(a)$. Making this substitution shows that for b not much larger than a, $f(b) = f(a) + f'(a)(b-a)$, approximately. To put this formula to computational use, the differential equation is used to express $f'(a)$ in terms of $f(a)$.

Example 3–1. To illustrate the method, consider the solution of the differential equation $C'(t) = C(t)$ satisfying the initial condition $C(0) = 1$. To apply the reasoning above in this context, a should be chosen as a point at which the value of the unknown function C is known. Since $C(0) = 1$ is given, $a = 0$ here. Next, b is selected as a slightly larger number at which the value of the function C is to be computed. Take $b = 0.1$ for illustration. The formula above then gives $C(0.1) = C(0) + C'(0)(0.1 - 0) = 1 + C'(0)(0.1)$, approximately. The computation would be complete if $C'(0)$ could be determined. The differential equation gives $C'(0) = C(0)$, and $C(0) = 1$ was given, so $C'(0) = 1$. Making this final substitution gives $C(0.1) = 1.1$, approximately.

Exercise 3–1. What is the exact solution to the equation $C'(t) = C(t)$ with $C(0) = 1$? What is the exact value of $C(0.1)$?

This computational method is called **Euler's Method**. Euler's Method can be used repeatedly to compute the approximate value of the solution at as many points as desried in order to sketch the approximate graph of the solution.

Example 3–2. In the previous example only two values were obtained, namely $C(0) = 1$ which was given, and $C(0.1) = 1.1$ approximately. The same idea can again be applied to obtain an approximate value for $C(0.2)$. Since $C(0.1)$ is now known, take $a = 0.1$ and $b = 0.2$ in the original formula to get $C(0.2) = C(0.1) + C'(0.1)(0.2 - 0.1) = 1.1 + C'(0.1)(0.1)$. The differential equation shows that $C'(0.1) = C(0.1)$ which was previously computed. The final approximation is $C(0.2) = 1.1 + (1.1)(0.1) = 1.21$.

This same idea can be used to compute the approximate value of the solution at any multiple of 0.1.

Exercise 3–2. Compute the approximate value of $C(t)$ when t is a multiple of 0.1 which lies between 0 and 1.

Euler's method is rather tedious to carry out by hand, but can be easily programmed into a computer.

Since Euler's method produces only approximate values for the solution, some estimate of the error in these values is useful. To discuss the error in a meaningful way, consider a more general scheme in which the approximate values of the solution are to be computed at multiples of a small positive number h. In the previous example, $h = 0.1$. Suppose the unknown function is $A(t)$, and that a first order differential equation for $A(t)$ is given. Using the Fundamental Theorem gives $A((n+1)h) = A(nh) + \int_{nh}^{(n+1)h} A'(s)\,ds$ as the exact relationship between the values of the solution at successive multiples of h. The approximation consists of replacing the integral in this expression with $A'(nh)h$, giving $A((n+1)h) = A(nh) + hA'(nh)$. This is an exact expression only if $A'(s)$ is constant. The differential equation is then used to write $A'(nh)$ in terms of $A(nh)$, producing the final formula of Euler's method. The error is therefore

$$\begin{aligned}
\text{error} &= \int_{nh}^{(n+1)h} A'(s)\,ds - \int_{nh}^{(n+1)h} A'(nh)\,ds \\
&= \int_{nh}^{(n+1)h} (A'(s) - A'(nh))\,ds \\
&= \int_{nh}^{(n+1)h} \int_{nh}^{s} A''(r)\,dr\,ds \\
&\le \max |A''(r)| h^2/2
\end{aligned}$$

by using the Fundamental Theorem and direct computation. Notice that this is the error on any *single* computation over an interval of length h. Since about t/h steps are required to pass from the starting point 0 to t, the error when using Euler's method to approximate $A(t)$ is less than $\max_{0\le r\le t} |A''(r)| th/2$. The magnitude of $A''(t)$ can often be usefully estimated from the differential equation.

Example 3–3. For the differential equation $A'(t) = A(t)$ with $A(0) = 0$, the error in using Euler's method to compute $A(1)$ would be $\max_{0\le r\le t} |A''(r)| h/2$ if the step size h is used. From the differential equation $A''(t) = A(t)$ too and $A(t)$ does not exceed 3 on this interval. So the error in approximating $A(1)$ is smaller than $3h/2$. The step size h can now be chosen to make the error as small as desired.

Exercise 3–3. How should h be selected if $A(1)$ is to be computed to within ± 0.001?

The error analysis reveals conditions under which Euler's method will perform poorly. If the second derivative of the solution is large over the interval of interest, then the step size must be extremely small in order to obtain an accurate estimate of the solution. Fully automated application of Euler's method will probably fail in such circumstances. It is wise to keep this consideration in mind when using computers to solve differential equations numerically.

A second aspect of Euler's method, and numerical methods in general, should also be kept in mind. Computers are able to manipulate numbers only with finite accuracy. While in theory Euler's method can be made as accurate as desired by using a small enough value of h, small values of h can lead to two computational problems. First, as h decreases, the number of iterations of Euler's method required to compute the approximate solution over a fixed interval increases. This leads to a corresponding increase in computation time. Second, due to round off error, the accuracy of the approximate solution will actually decrease as h shrinks below a certain level. As an extreme illustration, there is a non-zero value of h for which the computer will say $1 + h = 1$. In the example above, this choice of h would lead the computer to say that the constant function 1 is a solution of the equation. Computer generated solutions should always be viewed with a certain amount of skepticism!

The idea of using the differential equation as a relationship between the values of the derivative of the unknown function and the values of the function itself can be used to obtain the Maclaurin (or Taylor) series expansion of the solution. This provides a second method of approximating the values of the solution. Recall that the Maclaurin series for a function $f(x)$ is given by $f(x) = f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + \ldots$.

Example 3–4. The Maclaurin series for the solution of the equation $C'(t) = C(t)$ satisfying $C(0) = 1$ can be found as follows. Since $C(0) = 1$ is given, the first term in the Maclaurin series is known too. For the second term, $C'(0)$ is needed. Substituting $t = 0$ into the differential equation gives $C'(0) = C(0)$, so that $C'(0) = 1$ also. For the third term in the series, $C''(0)$ is needed. Differentiating both sides of the differential equation gives $C''(t) = C'(t)$. Substituting $t = 0$ here gives $C''(0) = C'(0) = 1$, by the previous step. The Maclaurin series for the solution is $C(t) = 1 + t + t^2/2 + \ldots$.

Exercise 3–4. What is $C'''(0)$?

This series method can be coupled with the idea underlying Euler's method in order to produce more accurate numerical approximations to the solution. Such modifications are called **Runge-Kutta methods**, and will not be discussed here. As was the case with Euler's method, the accuracy of a Runge-Kutta method depends on the size of the higher order derivatives of the solution.

Example 3–5. To carry out these computations using Maple, first issue the command `with(DEtools):` to initialize the software. The command

```
DEplot({diff(x(t),t) = x(t)},{x(t)},t = -3..3,[[x(0) = 1.0]],
x = -10..10,stepsize = .01,method = classical[foreuler]);
```

will draw the solution of the equation $x'(t) = x(t)$ with initial condition $x(0) = 1$ using Euler's method with a stepsize of 0.01. The commands

```
Order := 5;dsolve({diff(x(t),t) = x(t),x(0) = 1},x(t),type = series);
```

gives the first 5 terms of the Macluarin series solution of the same equation.

Problems

Problem 3–1. Use Euler's method with a step size $h = 0.1$ to approximate the solution of the equation $B'(t) = 5 - B(t)$ satisfying $B(0) = 1$ for $0 \le t \le 10$. How does your solution compare with the exact solution? What value of h should be used to keep the total error in $B(10)$ less than 0.001?

Problem 3–2. Use Euler's method to approximate the solution of the equation $C'(t) = (C(t))^2$ satisfying $C(0) = 1$ for $0 \le t \le 10$. What problems, if any, are encountered?

Problem 3–3. Find the first 4 terms of the Maclaurin series for the solution of the equation $D'(t) = tD(t)$ satisfying $D(0) = 1$.

Problem 3–4. Find the first 4 terms of the Taylor series about $t = 1$ for the solution of the equation $D'(t) = tD(t)$ satisfying $D(1) = 2$.

Solutions to Problems

Problem 3–1. The iterative formula obtained from Euler's method is $B(0.1n) = B(0.1(n-1)) + (0.1)(5 - B(0.1(n-1))) = 0.5 + 0.9B(0.1(n-1))$. The graph shows that $B''(t) = -B'(t) = B(t) - 5$ does not exceed 5 here, so the error in computing $B(10)$ does not exceed $25h$. So $h = 0.001/25$ should be used.

Problem 3–2. The second derivative $C''(t)$ is unbounded. The large error in using Euler's method is easily seen by comparing the approximate solution with the graph of the exact solution $1/(1-t)$.

Problem 3–3. From the differential equation, $D'(0) = 0$. Also $D''(t) = tD'(t) + D(t)$ and $D'''(t) = tD''(t) + 2D'(t)$. Hence $D''(0) = 1$ and $D'''(0) = 0$ so that $D(t) = 1 + t^2/2 + \ldots$.

Problem 3–4. The general form of the Taylor series about $t = 1$ is $D(t) = D(1) + D'(1)(t-1) + D''(1)(t-1)^2/2! + \ldots$. Now proceed as before.

Solutions to Exercises

Exercise 3–1. The exact solution is $C(t) = e^t$ so that $C(0.1) = e^{0.1} = 1.1051$ exactly.

Exercise 3–2. The same idea as before give $C(0.3) = 1.1C(0.2) = 1.331$, $C(0.4) = 1.1C(0.3) = 1.464$, and so on.

Exercise 3–3. Here $h = (2/3)(0.001) = 0.000667$. In practice, $h = 0.0005$ would probably be used.

Exercise 3–4. Using the differential equation gives $C'''(t) = C''(t)$ so that $C'''(0) = C''(0) = 1$.

§4. A Qualitative Method for First Order Autonomous Equations

The intuition behind Euler's method can also be used to develop a visual technique for studying the behavior of solutions of a differential equation. This method applied to first order autonomous equations is developed here.

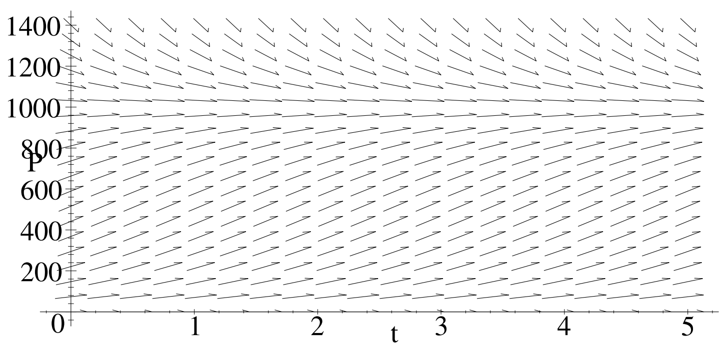
Example 4–1. The equation $P'(t) = 0.001P(t)(1000 - P(t))$ is the logistic equation of population growth. What are the basic properties of the solution? Even though the solution can be explicitly found, the basic properties of the solution can be difficult to determine from the final form of the solution.

Exercise 4–1. Find the general solution of this equation.

Euler's method was based on the observation that the differential equation gives a formula for the derivative of the unknown function in terms of the values of the unknown function. Instead of using the Fundamental Theorem of Calculus to develop a computational formula, recall instead that one interpretation of the derivative of a function is as the slope of graph of function at a point. Hence the differential equation gives a formula for the slope of the graph of the function in terms of the value of the function. This observation suggests the following graphical method. At each point $(t, P(t))$, use the differential equation to compute the slope $P'(t)$ of the solution which passes through this point. Represent this slope on the two dimensional grid by drawing an arrow at the point $(t, P(t))$ having slope $P'(t)$. The graph consisting of all of these arrows then gives the slope of the solution at any particular point. This arrow plot is called the **direction field plot** of the differential equation. The solution can then be graphed by following the arrows.

A computer can be easily programmed to do the calculations and plotting. The direction field for the equation $P'(t) = 0.001P(t)(1000 - P(t))$ is as follows.

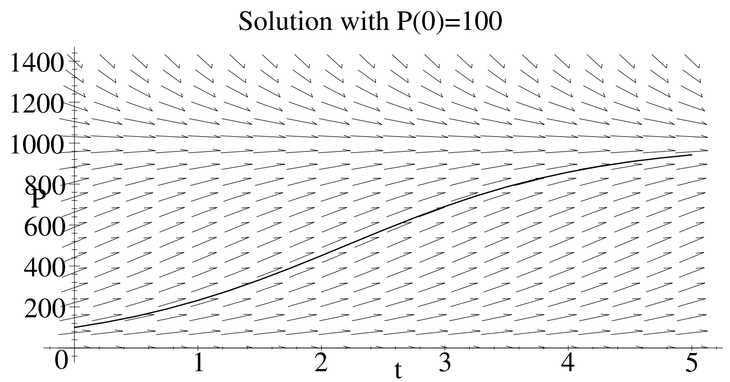
Direction Field Plot



To illustrate one specific computation for this equation, consider the point $(0, 1)$ in the graph of the function $P(t)$. This point corresponds to the solution taking the value $P(0) = 1$ at $t = 0$. The slope of the solution through this point is $P'(0) = 0.001(1)(1000 - 1) = 0.999$. So the arrow at the point $(0, 1)$ should have slope 0.999.

Exercise 4–2. What is the slope of the arrow at the point $(3, 10)$?

With the direction field in hand, any single solution can be graphed easily. For example, the solution satisfying $P(0) = 100$ is sketched by beginning at the point $(0, 100)$ and following the arrows.



Exercise 4–3. What is the graph of the solution satisfying $P(0) = 500$?

Exercise 4–4. What is the graph of the solution satisfying $P(0) = 2000$? What is $\lim_{t\to\infty} P(t)$ in this case?

As is seen in the preceding exercises, the direction field plot allows all solutions to be studied simultaneously. Features that are common to all solutions can then be identified. These common features often center around the constant function solutions. A reasonable first step in studying a differential equation is to find all solutions which are constant functions. These solutions are called the **equilibrium solutions**. For the equation of the previous example, equilibrium solutions are found by replacing $\frac{d}{dt}P(t)$ with 0 and then solving for $P(t)$. In the previous example, the constant functions which solve the equation are the zero solution and the function which always takes the value 1000.

Exercise 4–5. Verify that if $P(t) = 0$ for all $t > 0$ then $P(t)$ solves the equation. Do the same if $P(t) = 1000$ for all $t > 0$.

The constant function solution $P(t) = 1000$ is a **stable equilibrium**. This is because any solution which takes values near 1000 will approach 1000 in the limit as $t \to \infty$. The constant function solution $P(t) = 0$ is an **unstable equilibrium**. Any solution taking values near 0 (but not equal to 0) will tend in the limit as $t \to \infty$ to either 1000 or $-\infty$, not to 0.

Exercise 4–6. Is it always true that the direction field for an autonomous equation is the same along any two vertical lines?

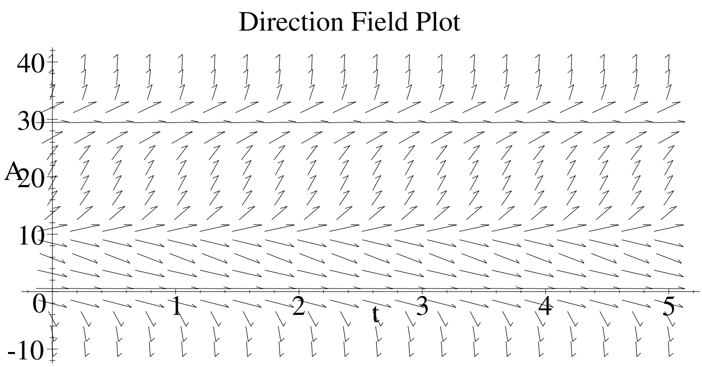
Example 4–2. The Maple command to plot the direction field for the equation $x'(t) = x(t)$ is

```
DEplot({diff(x(t),t) = x(t)},{x(t)},t = -3..3,x = -10..10);.
```

Problems

Problem 4–1. For the population model $\frac{d}{dt}P(t) = 5P(t)(1000 - P(t))$ with $P(0) = 100$ find the asymptotic population size $\lim_{t\to\infty} P(t)$.

Problem 4–2. Suppose a first order autonomous equation has direction field



What are the equilibrium solutions? Which of the equilibrium solutions are stable? What is the asymptotic behavior of a solution satisfying $A(0) = 1$? Satisfying $A(0) = 15$? Give one equation for which this plot could be the direction field.

Problem 4–3. Suppose $a > 0$ and $b > 0$ are constants and $\frac{d}{dt}P(t) = aP(t) - b(P(t))^2$ for $t > 0$. Describe the behavior of a solution which has a positive value at time $t = 0$. Your answer will depend on a and b.

Problem 4–4. Suppose $\frac{d}{dt}x(t) = (1 - x(t))(2 - x(t))(3 - x(t))$ for $t > 0$. Sketch the direction field for this equation. What is the asymptotic behavior of the solution which satisfies $x(0) = 3/2$? What if $x(0) = 4$? If $x(0) = -5$?

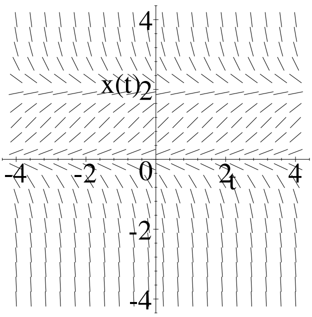
Problem 4–5. For each of the equations below, give the letter corresponding to its direction field plot. Also compute the indicated limit if $x(0) = 0$.

(a) For the equation $x'(t) = x(t) - 2$ the direction field plot is ______ and $\lim_{t\to\infty} x(t) =$

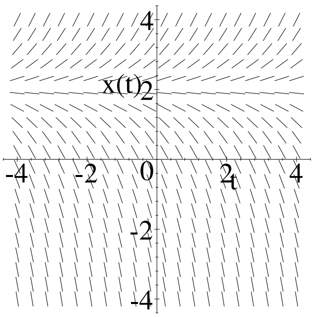
(b) For the equation $x'(t) = x(t)^2 - 4$ the direction field plot is ______and $\lim_{t\to\infty} x(t) =$

(c) For the equation $x'(t) = x(t)^3 - 4x(t)$ the direction field plot is ______ and $\lim_{t\to\infty} x(t) =$

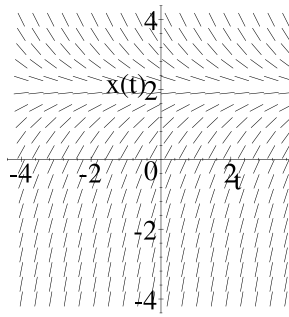
Plot A



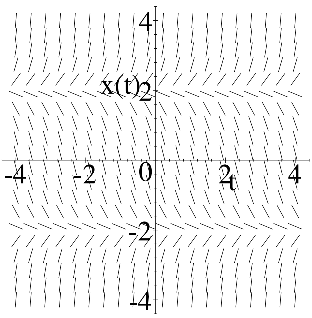
Plot B



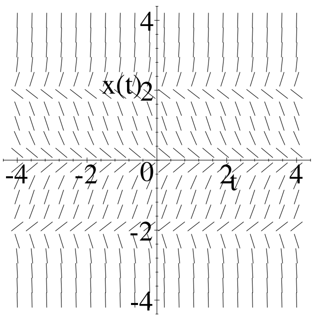
Plot C



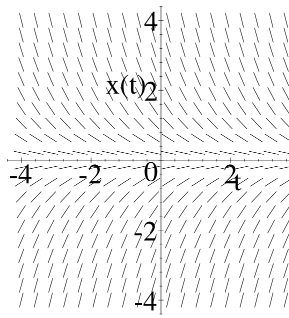
Plot D



Plot E



Plot F



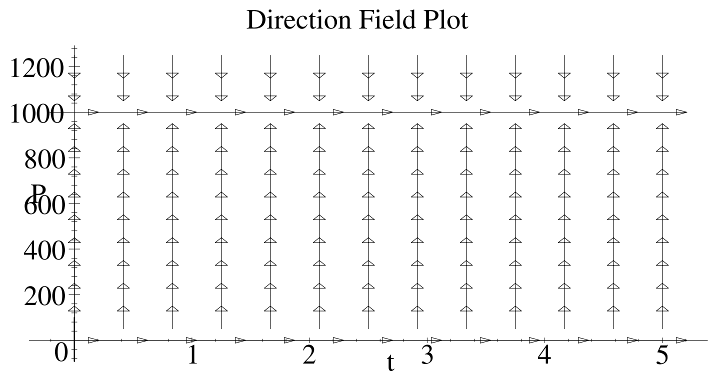
Problem 4–6. Solve the equation $A'(t) = 1 + A(t)$ with the initial condition $A(0) = 0$. What is $\lim_{t\to\infty} A(t)$?

Problem 4–7. Can the function $x(t) = \sin t$ be a solution of the equation $\frac{d}{dt}x(t) = 1 - (x(t))^2$? Why or why not? Hint: What is the direction field?

Problem 4–8. The velocity of a falling body of mass m in the presence of air resistance was earlier found to satisfy the equation $m\frac{d}{dt}v(t) = -kv(t) - mg$. Plot the direction field for $v(t)$. What are the equilibrium solutions, and which are stable? What is the asymptotic velocity of the falling body?

Solutions to Problems

Problem 4–1. The equilibrium solutions are again 0 and 1000. The direction field is

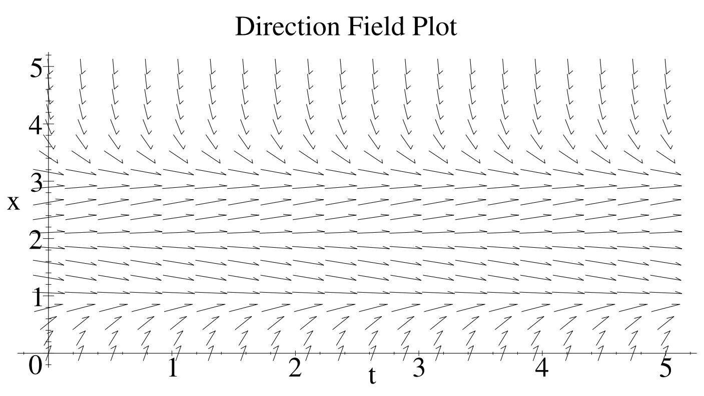


Thus the asymptotic population size is 1000.

Problem 4–2. The equilibrium solutions are 0, 10, and 100 and none are stable. A solution with $A(0) = 1$ will have asymptotic value 0; a solution with $A(0) = 15$ will have asymptotic value 100. One such equation is $A'(t) = A(t)^2(A(t)-10)(A(t)-30)^2$.

Problem 4–3. In this case the equilibrium solutions are 0 and a/b, and a/b is a stable equilibrium. The asymptotic value of a solution which has a postive value at $t = 0$ is therefore a/b.

Problem 4–4. The direction field is



From the diagram 1 and 3 are stable equilibria, while 2 is unstable. If $x(0) = 3/2$ the asymptotic value is 1; if $x(0) = 4$ the asymptotic value is 3; if $x(0) = -5$ the asymptotic value is 1.

Problem 4–5. In (a) the direction field plot is B and the limit is $-\infty$; in (b) the direction field plot is D and the limit is -2; in (c) the direction field plot is E and the limit is 0.

Problem 4–6. Separation of variables gives $\dfrac{A'(t)}{1+A(t)} = 1$ from which integration gives $\ln|1 + A(t)| = t + C$. The condition $A(0) = 0$ gives $C = 0$ and also allows the removal of the absolute value signs to give $A(t) = e^t - 1$. From the solution or from the direction field, $\lim_{t\to\infty} A(t) = \infty$.

Problem 4–7. The direction field shows that 1 is a stable equilibrium while -1 is not. Hence the proposed solution $\sin t$ would have to have asymptotic value 1, which precludes oscillation.

Problem 4–8. The equilibrium solution is $-mg/k$, which is stable. The asymptotic velocity is therefore $-mg/k$.

Solutions to Exercises

Exercise 4–1. By separation of variables $|P(t)/(1000 - P(t))| = e^{t+C}$. How are the absolute value signs removed? What are the constant solutions of the equation?

Exercise 4–2. The slope is $P'(3) = 0.001P(3)(1000-P(3)) = 0.001(10)(1000-10) = 9.90$.

Exercise 4–3. Start at the point $(0, 500)$ and follow the arrows. What is $\lim_{t\to\infty} P(t)$ in both of these cases?

Exercise 4–4. The limit is still 1000.

Exercise 4–5. Just substitute these constant solutions in and verify that the equation holds.

Exercise 4–6. Yes. Since the equation is autonomous, the formula for the derivative does not depend on the independent variable alone.

§5. The General First Order Linear Equation

The integrating factor method of solving the general first order linear equation is discussed.

Example 5–1. A rocket is launched at time $t = 0$ and its engine provides a constant thrust for 10 seconds. During this time the burning of the rocket fuel constantly decreases the mass of the rocket. The problem is to determine the velocity $v(t)$ of the rocket at time t during this initial 10 second interval. Denote by $m(t)$ the mass of the rocket at time t and by U the constant upward thrust (force) provided by the engine. Applying Newton's Law gives

$$\frac{d}{dt}(m(t)v(t)) = U - kv(t) - m(t)g$$

where an air resistance term is included in addition to the gravitational and thrust terms. This equation is linear, non-homogeneous, and not autonomous. How is this equation solved?

The method is called the **integrating factor technique**.

(1) Re-arrange the equation as necessary to be in the form $\frac{d}{dt}v(t) + a(t)v(t) = b(t)$. The coefficient of the derivative term must be 1.

(2) Compute the **integrating factor** $e^{\int a(t)\,dt}$. The integral here can use *any* antiderivative of $a(t)$.

(3) Multiply both sides of the equation obtained in step 1 by the integrating factor. The result is guaranteed to be in the form

$$\frac{d}{dt}\left(v(t)e^{\int a(t)\,dt}\right) = b(t)e^{\int a(t)\,dt}.$$

(4) Integrate both sides and solve for the unknown function. Use the initial condition and the Fundamental Theorem of Calculus.

To illustrate the method of solution the simpler equation

$$\frac{d}{dt}v(t) = tv(t) + t$$

or

$$\frac{d}{dt}v(t) - tv(t) = t$$

with initial condition $v(0) = 0$ is studied first. The integrating factor here is $e^{\int -t\,dt}$. Multiply both sides of the equation by this factor. After multiplication the left side of the equation will be the derivative of a product of the integrating factor and the unknown function $v(t)$. This gives the equation

$$\frac{d}{dt}\left(e^{-t^2/2}v(t)\right) = te^{-t^2/2}.$$

Both sides can now be integrated with respect to t from 0 to s. Using the Fundamental Theorem of Calculus on the left hand side, this gives

$$e^{-s^2/2}v(s) - e^{0^2/2}v(0) = \int_0^s e^{-t^2/2}\,dt.$$

The integral on the right side can be integrated by substitution to give $1 - e^{-s^2/2}$. Using this fact and the initial condition $v(0) = 0$, the equation above can now be solved to give

$$v(s) = e^{s^2/2}(1 - e^{-s^2/2}) = e^{s^2/2} - 1.$$

Exercise 5–1. Verify carefully all of the steps in this derivation.

The integrating factor technique will solve any first order linear equation, and thus provides yet another method of solving first order linear equations with constant coefficients and also certain autonomous equations. The success of the method depends on the ability to compute the two integrals that arise. Since the second integral is a definite integral, numerical methods can be used to find the value of the solution at any particular point.

Example 5–2. The equation $\frac{d}{dt}v(t) = tv(t)+t^2$ with initial condition $v(0) = 5$ will be solved. First, the equation is rewritten as $\frac{d}{dt}v(t)-tv(t) = t^2$ and the integrating factor $e^{-t^2/2}$ is found. Multiplying by the integrating factor gives $\frac{d}{dt}\left(e^{-t^2/2}v(t)\right) = t^2e^{-t^2/2}$. Now integrate both sides from 0 to s to obtain

$$e^{-s^2/2}v(s) - 5e^{-0^2/2} = \int_0^s t^2e^{-t^2/2}\,dt$$

or

$$v(s) = 5 + e^{s^2/2}\int_0^s t^2e^{-t^2/2}\,dt.$$

This last expression can be used together with Simpson's Rule, or another numerical integration technique, to numerically compute the value of v at any particular s.

Example 5–3. As the final example, the general solution of the general first order linear equation will be found. As above, the equation is first written as $v'(t) + a(t)v(t) = b(t)$ and the initial condition is assumed to be the value $v(t_0)$ for some point t_0. To make the computations simpler, the integrating factor is taken as the definite integral $e^{\int_{t_0}^t a(r)\,dr}$. Multiplying by the integrating factor transforms the original equation into $\left(e^{\int_{t_0}^{t} a(r)\,dr} v(t)\right)' = b(t)e^{\int_{t_0}^{t} a(r)\,dr}$. Using the Fundamental Theorem and the initial condition to integrate both sides of this equation from 0 to s gives $e^{\int_{t_0}^{s} a(r)\,dr} v(s) - v(t_0) = \int_{t_0}^{s} e^{\int_{t_0}^{t} a(r)\,dr} b(t)\,dt$, which is finally rearranged to give

$$v(s) = v(t_0)e^{-\int_{t_0}^{s} a(r)\,dr} + \int_{t_0}^{s} b(t)e^{-\int_{t}^{s} a(r)\,dr}\,dt.$$

Although this expression is complicated, the connection between the initial condition, the function $b(t)$, and the solution $v(s)$ is clearly displayed.

Problems

Problem 5–1. What is the general solution of the equation $\frac{d}{dt}x(t) = 2tx(t) + t$? Hint: Write the initial condition as $x(t_0) = x_0$.

Problem 5–2. What is the solution of $\frac{d}{dt}y(t) - 3y(t) = e^t$ which also satisfies $y(0) = -3$?

Problem 5–3. Consider the equation $C'(t) = 5C(t)(7 - C(t))$. What *constant functions* are solutions of the equation? If $C(0) = 3$ and $C(t)$ is a solution of the equation, what is $\lim_{t\to\infty} C(t)$?

Problem 5–4. Find the solution of the equation $B'(t) = B(t) + e^{2t}$ which satisfies $B(0) = 1$. What is $\lim_{t\to\infty} B(t)$?

Problem 5–5. If $m(t) = 11 - t$, $U = 200$, and $k = 0$ the equation of motion of the rocket is $\frac{d}{dt}((11 - t)v(t)) = 200 - (11 - t)g$. Find $v(t)$ for $0 \le t \le 10$. Assume $v(0) = 0$. Make a graph of the velocity as a function of time.

Problem 5–6. If $m(t) = 11 - t$, $U = 200$, and $k = 2$ the equation of motion of the rocket is $\frac{d}{dt}((11 - t)v(t)) = 200 - 2v(t) - (11 - t)g$. Find $v(t)$ for $0 \le t \le 10$ assuming $v(0) = 0$. Make a graph of the velocity as a function of time.

Problem 5–7. A brick initially at temperature 20 is placed in an oven whose temperature at time t is $70 + 5t$. Find the temperature of the brick at time t. Assume the proportionality constant in Newton's Law of Cooling is 1.

Problem 5–8. Some savings accounts have an interest rate that increases with the age of the account. Suppose your account has an initial interest rate of 5% which increases linearly to 8% after 3 years. Suppose your initial rate of deposit into the account is 1000 per year and this increases by 100 annually. What is your account balance after 2 years? How much money have you contributed in the first 2 years, and how much interest have you earned? State any additional assumptions you make.

Problem 5–9. Fresh water enters a swimming pool at the rate of 500 gallons per hour, and water leaves through the pool drain at the same rate. The pool holds a total of 50,000 gallons. A quarter pound chlorine tablet placed in the pool will dissolve completely in 5 days. How many quarter pound tablets should be placed in a chlorine free pool so that 2 pounds of dissolved chlorine will be in the pool at the end of 3 days?

Problem 5–10. Garfield the cat is on a diet. His basic metabolism consumes 200 calories per day. His exercise program consumes 100 calories per day per kilogram of body mass. Food intake provides 750 calories per day. Caloric intake that is not consumed by basic metabolism or exercise is converted into fat; calories needed for basic metabolism or exercise in excess of caloric intake are obtained from the fat store. Body fat stores or releases calories at a rate of 5,000 calories per kilogram of fat. Assume that the conversion of calories to fat (and vice-versa) is perfectly efficient. Denote by $G(t)$ the mass, in kilograms, of Garfield on day t of this diet regimen. Write a differential equation for $G(t)$ using the information given. If Garfield sticks religiously to this diet, what is his ultimate mass?

Solutions to Problems

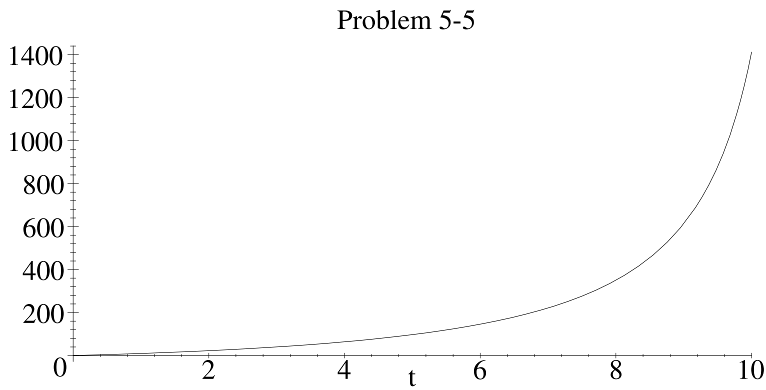
Problem 5–1. Using the general form of the last example with $a(t) = -2t$ and $b(t) = t$ gives the general solution as $x(s) = x_0 e^{-s^2+t_0^2} + \int_{t_0}^{s} te^{s^2-t^2}\, dt = (x(t_0) + 1/2)e^{s^2-t_0^2} - 1/2$. This equation is also separable.

Problem 5–2. The integrating factor is e^{-3t} so that finally $y(t) = (-5/2)e^{3t} - (1/2)e^{t}$.

Problem 5–3. The constant function solutions are the constant function 0 and the constant function 7. If $0 < C(t) < 7$, $C'(t) > 0$ and so the function $C(t)$ increases toward the value 7. Hence $\lim_{t\to\infty} C(t) = 7$.

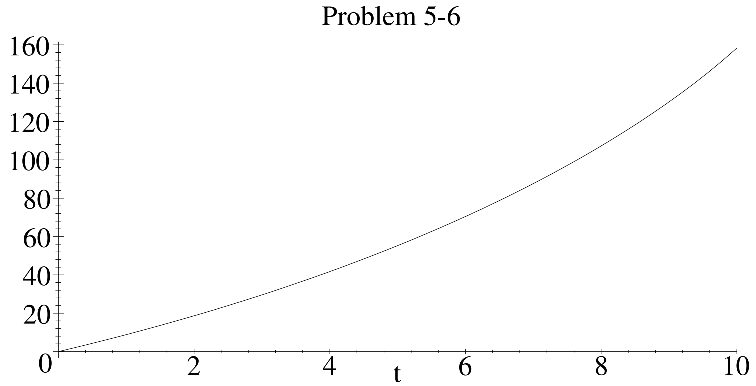
Problem 5–4. Rewriting the equation as $B'(t) - B(t) = e^{2t}$ shows that the integrating factor is e^{-t}, so that after multiplying by the integrating factor the equation becomes $(e^{-t}B(t))' = e^{t}$. Integration from 0 to t gives $e^{-t}B(t) - B(0) = e^{t} - 1$ so that $B(t) = e^{2t}$ after using the initial condition. The limit is infinity.

Problem 5–5. The equation to be solved is $\frac{d}{dt}((11-t)v(t)) = 200 - (11-t)g$. Since the right side does not depend on $v(t)$, the equation can be solved by simple integration. The final solution as $v(t) = \dfrac{200t + (g/2)(11-t)^2 - 121g/2}{11-t}$ for $0 \le t \le 10$.



Problem 5–6. Here the equation is $\frac{d}{dt}((11-t)v(t)) = 200 - (11-t)g - 2v(t)$ with $v(0) = 0$. Expanding the derivative on the left hand side and rearranging terms gives $\frac{d}{dt}v(t) + \frac{1}{11-t}v(t) = 200/(11-t) - g$. The integrating factor is thus $1/(11-t)$. Multiplying by the integrating factor gives $\frac{d}{dt}\left(\frac{1}{11-t}v(t)\right) = 200/(11-t)^2 - g/(11-t)$. Integrating and solving gives $v(t) = 200 + (11 -$

$t)g\ln(11-t)-(200/11+g\ln(11))(11-t)$.



Problem 5–7. Newton's Law of Cooling gives the equation $\frac{d}{dt}B(t) = (70 + 5t - B(t))$ as the equation to be solved with $B(0) = 20$. The integrating factor is e^t and so $e^tB(t) - 20 = \int_0^t (70+5s)e^s\,ds$. Using integration by parts gives $\int se^s\,ds = e^s(s-1)$, hence $e^tB(t) - 20 = 70e^t - 70 + 5((t-1)e^t + 1)$ or $B(t) = 65 + 5t - 45e^{-t}$.

Problem 5–8. The differential equation for the amount $A(t)$ in the account is $\frac{d}{dt}A(t) = \frac{5+t}{100}A(t) + 1000 + 100t$ with $A(0) = 0$. The contributions $C(t)$ up to time t satisfies $\frac{d}{dt}C(t) = 1000 + 100t$ with $C(0) = 0$. The C equation is solved easily by integration to give $C(t) = 1000t + 50t^2$ and so $C(2) = 2200$. The integrating factor for the A equation is $e^{-(5+t)^2/200}$. Using definite integrals from 0 to s then gives $A(s) = e^{(5+s)^2/200}\int_0^s (1000+100t)e^{-(5+t)^2/200}\,dt$. Plugging in $s = 2$ and integrating numerically gives $A(2) = 2,340.99$.

Problem 5–9. Let $C(t)$ be the number of pounds of dissolved chlorine in the pool after t days. Assume that a tablet dissolves at a constant rate. This means that each tablet supplies chlorine at a rate of 1/20 pound per day for $0 \le t \le 5$. Also assume that the dissolved chlorine is instantly mixed throughout the pool. In one day, $500 \times 24 = 12,000$ gallons of mixture leaves the pool and is replaced by fresh water. Hence $C'(t) = N/20 - (12,000/50,000)C(t)$ where N is the number of quarter pound tablets. Since $C(0) = 0$, this equation can be solved and the value of N required so that $C(3) = 2$ can then be found.

Problem 5–10. Computing in terms of calories gives the equation $5000G'(t) = 750 - 200 - 100G(t)$. The constant function solution of the differential equation is $G(t) = 5.5$ kilograms, which from the direction field is the ultimate mass for Garfield.

§6. A Qualitative Method for First Order Equations

The qualitative method developed earlier for autonomous equations is extended to first order equations.

For non-autonomous equations the derivative depends directly on the independent variable as well as the solution. The direction field can still be plotted, but the possible behavior of the solutions can be more complicated than before.

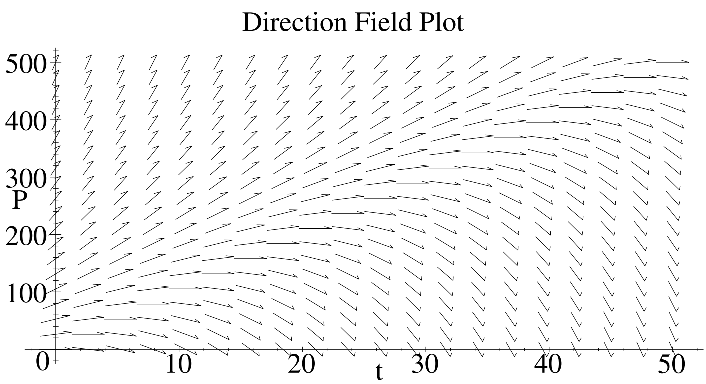
Example 6–1. Suppose a population with unlimited food supply is also subject to (net) emigration. If the emigration is at a variable rate one possible model could be

$$\frac{d}{dt}P(t) = 0.1P(t) - t.$$

The objective is to study the solutions of this equation without actually finding the formula for the solution.

Exercise 6–1. What is the exact solution of this equation?

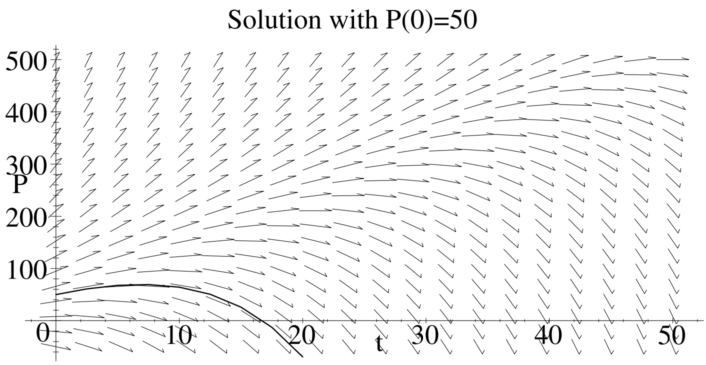
The direction field for this equation is as follows.



As before, the qualitative behavior of the solution with given initial value can be sketched from the direction field by following the arrows.

Exercise 6–2. Sketch the solution satisfying the initial condition $P(0) = 200$.

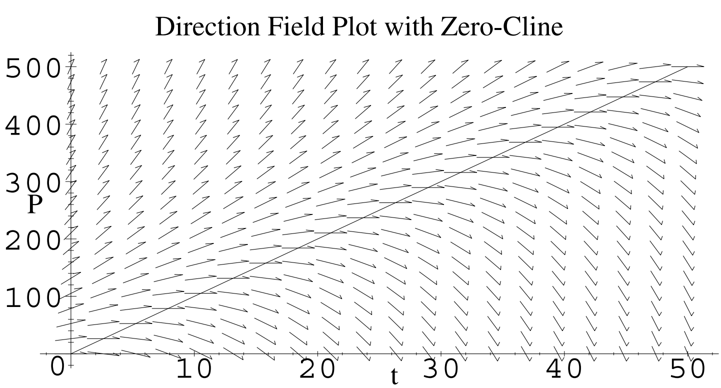
A more careful examination of the direction field shows that the long run behavior of the solution depends on the intial population size. For example, the solution with $P(0) = 50$ is graphed as follows.



The graph clearly shows that the population becomes extinct in this case.

In the case of autonomous equations, a central role was played by constant function solutions. These equilibrium solutions were used to determine the limiting behavior of a solution of the equation from its intial value. As is the case for the present equation, non-autonomous equations usually do not have constant function solutions. For these equations the role of the equilibrium solution is played graphically by the **zero-cline**, which is the curve along which the slope of the solution is zero. It is important to keep in mind that the zero-cline is usually *not* a solution of the equation.

For the equation $P'(t) = 0.1P(t) - t$ the zero-cline is the curve $P(t) = 10t$. The graph of the direction field and the zero-cline is as follows.



The zero-cline always divides the direction field plot into two regions. In one region the direction field has positive slope; in the other region the direction field has negative slope. In this particular case the region of positive slope is above the zero-cline while the region of negative slope is below the zero-cline. Simple reasoning from the differential equation shows that any solution which starts at

$t = 0$ with a slope exceeding 10 will never cross the zero-cline. Such a solution will tend to infinity. A solution starting at time $t = 0$ with a slope of less than 10 will eventually cross the zero-cline. Once this occurs, the solution will diverge to minus infinity. This analysis shows that the initial population size must be at least 100 if the population is to avoid extinction.

Exercise 6–3. Verify the conclusion of the graphical analysis by examining the behavior of the exact solution obtained earlier.

It is not always possible to perform such a detailed graphical analysis in the general case. Usually at least some aspects of the behavior of the solutions can be obtained using these simple graphical methods.

Problems

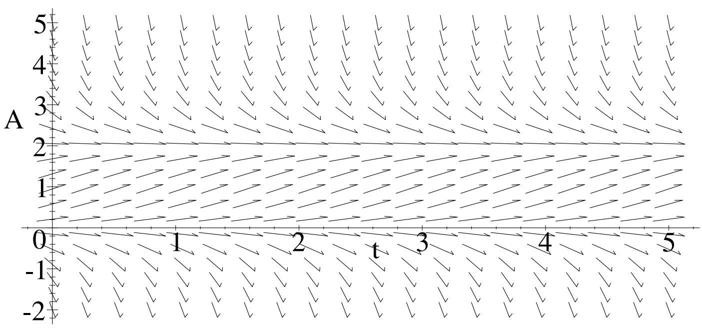
Problem 6–1. What characteristic of a direction field will identify it as the direction field of an autonomous equation?

Problem 6–2. The function $A(t)$ satisfies a first order differential equation which has direction field A below. Sketch the solution satisfying the initial condition $A(0) = 5$ for $0 \leq t \leq 4$.

Problem 6–3. The function $B(t)$ satisfies a first order differential equation which has direction field B below. Sketch the solution satisfying the initial condition $B(0) = 3$ for $0 \leq t \leq 4$.

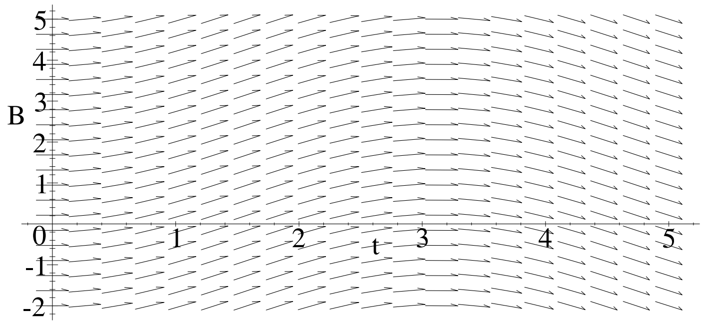
Problem 6–4. Could direction field A below come from the differential equation $\frac{d}{dt}A(t) = A(t)(2 - A(t))$? Why?

Direction Field Plot A



Problem 6–5. Could direction field B below come from the differential equation $\frac{d}{dt}B(t) = \sin B(t)$? Why?

Direction Field Plot B



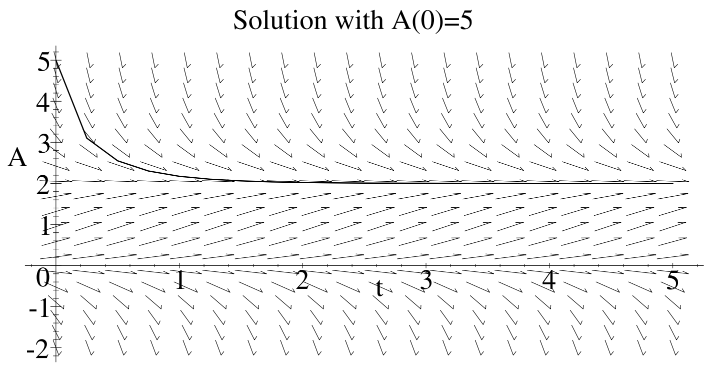
Problem 6–6. Consider the equation $x'(t) - x(t) = t$. Make a rough sketch of the direction field, showing the isocline corresponding to zero slope and indicating

where the direction field has positive and negative slope. If $x(0) = 3$, what is $\lim_{t\to\infty} x(t)$? If $x(0) = -3$, what is $\lim_{t\to\infty} x(t)$?

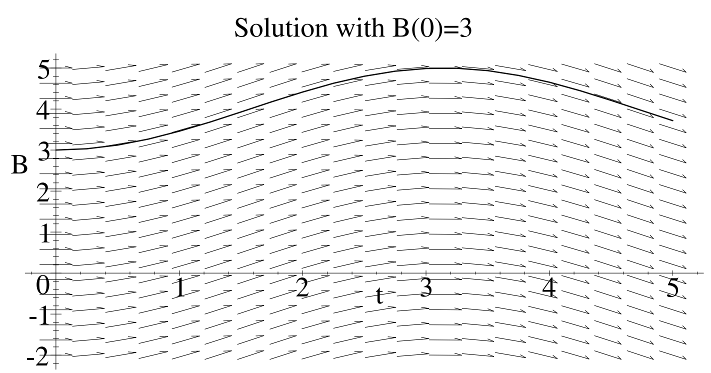
Solutions to Problems

Problem 6–1. Since for an autonomous equation the derivative of the function does not depend on the independent variable, the direction field will not depend on the independent variable either.

Problem 6–2.



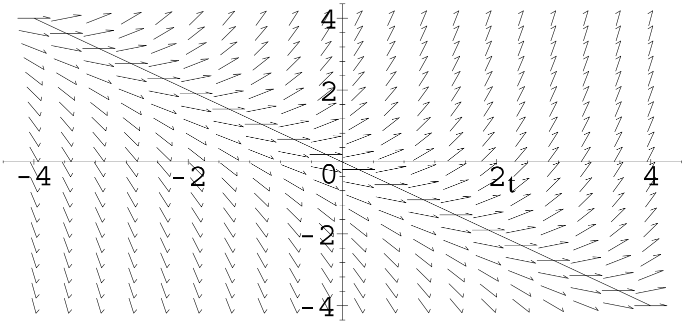
Problem 6–3.



Problem 6–4. Direction field A has 2 equilibrium solutions, one at 0 and the other at 2. The equilibrium solution at 2 is stable while the solution at 0 is unstable. The sign of the derivative as given in plot A as well as the equilibrium solutions agree with those of the given equation. The answer is **yes**.

Problem 6–5. Direction field plot B does not depend on the value of $B(t)$ since the field lines are all parallel. The answer is **no**. It is more likely that plot B corresponds to the equation $\frac{d}{dt}B(t) = \sin t$.

Problem 6–6.

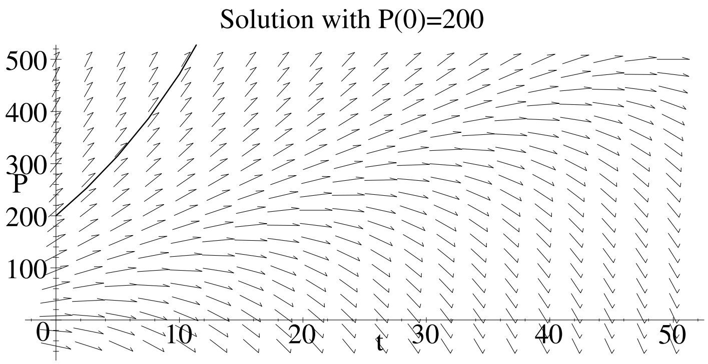


From the direction field, if $x(0) = 3$ then $\lim_{t\to\infty} x(t) = \infty$. From the direction field $\lim_{t\to\infty} x(t) = -\infty$ in the case $x(0) = -3$.

Solutions to Exercises

Exercise 6–1. Using the integrating factor $e^{-t/10}$ after rewriting the equation shows that the general solution is $P(t) = 10t + 100 + Ce^{t/10}$.

Exercise 6–2. Simply follow the arrows, keeping in mind that the arrows are *tangent* to the solution curve at each point. The graph should be as follows.



Exercise 6–3. Writing the solution found earlier by evaluating the unknown constant in terms of the initial population size gives $P(t) = 10t + 100 + (P(0) - 100)e^{t/10}$. Hence if $P(0) > 100$, the population size increases indefinitly, while if $P(0) < 100$, extinction occurs.

§7. Homogeneous Second Order Linear Equations

The general method of solving a homogeneous second order linear equation with constant coefficients is presented.

Example 7–1. A spring has one end attached to a vertical wall and the other end attached to a 1 kilogram mass. The mass lies on a horizontal surface and $x(t)$ denotes the displacement from equilibrium at time t. At time 0, the displacment is 1 unit and the mass is released. According to Hooke's Law, the force on the mass exerted by the spring is proportional to the displacement from equilibrium. The proportionality constant is called the **spring constant** which measures the stiffness of the spring. Assume the spring constant is 3. Assume the frictional force is zero and the air resistance is equal in magnitude to four times the velocity. Using Newton's Law then gives the equation

$$1 \cdot \frac{d^2}{dt^2}x(t) = -4\frac{d}{dt}x(t) - 3x(t)$$

as the equation of motion for $t > 0$.

The equation derived in the example is a second order linear homogeneous equation with constant coefficients. In order to solve the equation the technique used earlier in the first order case is adapted, as follows. Try a solution of the form $x(t) = e^{mt}$, where m is to be determined. Plugging this trial solution into the equation gives

$$m^2 e^{mt} = -4me^{mt} - 3e^{mt}$$

and this equation will hold for all $t > 0$ only if $m^2 = -4m - 3$. The equation m must satisfy, here $m^2 + 4m + 3 = 0$, is called the **characteristic equation**. The values of m that satisfy the characteristic equation in this case are $m = -3$ and $m = -1$. The corresponding trial solutions are e^{-3t} and e^{-t}. The **superposition principle** then states that the general solution is $x(t) = C_1e^{-3t} + C_2e^{-t}$. Here C_1 and C_2 are two arbitrary constants which are determined from the initial conditions.

Exercise 7–1. Verify that $C_1e^{-3t} + C_2e^{-t}$ solves the equation no matter what values C_1 and C_2 have.

In this case the initial displacement is 1 and the initial velocity may be assumed to be 0. This means that $x(0) = 1$ and $\frac{d}{dt}x(t)\Big|_{t=0} = 0$. Using these conditions and the general form of the solution gives

$$C_1 + C_2 = 1$$
$$-3C_1 - C_2 = 0$$

from which $C_1 = -1/2$ and $C_2 = 3/2$. Hence $x(t) = -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t}$ is the displacement at time t.

Exercise 7–2. Describe in words the motion of the mass in this case. Which force is stronger, the spring or air resistance? Make a graph of the displacement $x(t)$ for $0 \le t \le 5$.

The method for solving second order linear homogeneous equations with constant coefficients is as follows.

(1) Try a solution of the form e^{mt}. This produces the characteristic equation (a quadratic equation) to be solved for m.

(2) Solve the characteristic equation.

(3) The general solution of the differential equation is obtained from the roots of the characteristic equation as follows.

Roots of the Characteristic Equation	General Solution
2 Distinct Real Roots a and b	$C_1 e^{at} + C_2 e^{bt}$
Conjugate Complex Roots $a \pm ib$	$e^{at}(C_1 \cos bt + C_2 \sin bt)$ **or:** $C_1 e^{at} \cos(bt + C_2)$
Repeated Real Root a	$e^{at}(C_1 + C_2 t)$

(4) Use the initial conditions to determine the unknown constants in the general solution.

The alternate form in the conjugate complex roots case is referred to as the **phase-amplitude form**. This form is useful in certain cases.

Example 7–2. As a second example, suppose air resistance was neglected. The equation of motion then becomes

$$\frac{d^2}{dt^2}x(t) = -3x(t).$$

The characteristic equation is then $m^2 = -3$ which has conjugate complex roots $m = \pm i\sqrt{3}$. From the table above, $x(t) = C_1 \cos(t\sqrt{3}) + C_2 \sin(t\sqrt{3})$ is the general solution, where C_1 and C_2 are arbitrary constants. The values of these constants are determined from the initial conditions of the problem.

Exercise 7–3. Verify that $C_1 \cos(t\sqrt{3}) + C_2 \sin(t\sqrt{3})$ does in fact solve the equation.

If the initial conditions are $x(0) = 1$ and $x'(0) = 0$, as before, the two constants C_1 and C_2 must satisfy

$$C_1 = 1$$
$$C_2\sqrt{3} = 0$$

which leads to $C_1 = 1$ and $C_2 = 0$. The position is then given by $x(t) = \cos(t\sqrt{3})$ in this case.

Exercise 7–4. Check carefully the computation of the constants C_1 and C_2 in this example.

Exercise 7–5. Make a graph of the displacement $x(t)$ for $0 \le t \le 5$.

Exercise 7–6. Show that $e^{it\sqrt{3}}$ and $e^{-it\sqrt{3}}$ solve the differential equation above. Since **Euler's Formula** states that $e^{it\sqrt{3}} = \cos(t\sqrt{3}) + i\sin(t\sqrt{3})$ and $e^{-it\sqrt{3}} = \cos(t\sqrt{3}) - i\sin(t\sqrt{3})$ the two real valued functions $\cos(t\sqrt{3})$ and $\sin(t\sqrt{3})$ must also be solutions. This is how the table entry was derived.

So far the cases in which the characteristic equation has two unequal real roots and conjugate complex roots have been discussed. The case in which the characteristic equation has a repeated real root remains.

Example 7–3. Consider the equation $\frac{d^2}{dt^2}x(t) + 4\frac{d}{dt}x(t) + 4x(t) = 0$. The characteristic equation is then $m^2 + 4m + 4 = 0$ which has a repeated real root at $m = -2$. Using the table gives $x(t) = C_1te^{-2t} + C_2e^{-2t}$ as the general solution.

Exercise 7–7. Use the initial conditions to find the constants C_1 and C_2 in this case. Graph the solution for $0 \le t \le 5$.

Exercise 7–8. How do the solutions in the 3 different cases compare?

To see how the general solution in the case of repeated real roots is derived, first notice that e^{-2t} is one solution of the original equation. Define a new function $y(t)$ by the formula $y(t) = \frac{d}{dt}x(t) + 2x(t)$. Then the original equation for $x(t)$ shows that $y(t)$ must satisfy the first order equation

$$\frac{d}{dt}y(t) = -2y(t).$$

Hence $y(t) = C_1e^{-2t}$ from the discussion of simple first order equations. Now recalling the definition of $y(t)$ gives the equation

$$\frac{d}{dt}x(t) + 2x(t) = C_1e^{-2t}$$

which can be solved to give $x(t) = C_1te^{-2t} + C_2e^{-2t}$ as the general solution.

Exercise 7–9. Show that $A\cos t + B\sin t = \sqrt{A^2 + B^2}\cos(t + \phi)$ where ϕ satisfies $\cos\phi = A/\sqrt{A^2 + B^2}$ and $\sin\phi = -B/\sqrt{A^2 + B^2}$. Use this fact to verify the correctness of the phase amplitude form.

Problems

Problem 7–1. Find the position of the mass as a function of time if the initial displacement is 0 and the initial velocity is 1. Assume that the surface is frictionless, there is no air resistance, and the spring constant is 4.

Problem 7–2. Suppose the surface is frictionless and the air resistance force has magnitude equal to twice the velocity. Assume the spring constant is 1. Write the equation of motion of the mass. Solve the equation if the initial displacement is 1 and the initial velocity is 0.

Problem 7–3. True or False: The equation $A'(t) - 3A(t) = t$ is a first order homogeneous linear equation.

Problem 7–4. Find the general solution of the equation $\frac{d^2}{dt^2}x(t) = 0$.

Problem 7–5. Find the solution of the equation $A'(t) = t^2 A(t)$ which satisfies $A(0) = 1$.

Problem 7–6. The temperature $T(x)$ at position x along a rod satisfies the differential equation $\frac{d^2}{dx^2}T(x) + 4T(x) = 0$. Find the temperature distribution in the rod if $T(0) = 0$ and $T(2) = 0$. (The conditions $T(0) = 0$ and $T(2) = 0$ are called **boundary conditions**. In this context the boundary conditions imply that the ends of the rod are being held at a fixed temperature of 0.)

Problem 7–7. Find all values of the constant V so that the equation $\frac{d^2}{dx^2}T(x) + VT(x) = 0$ has a solution which is not 0 for all x and yet satisfies the boundary conditions $T(0) = 0$ and $T(2) = 0$. What is the general form of these solutions?

Problem 7–8. Solve the equation $\dfrac{d^2}{dt^2}A(t) = 6t + 8$ with $A(0) = 4$ and $A'(0) = 9$. Hint: Integrate!

Solutions to Problems

Problem 7–1. The differential equation is $\frac{d^2}{dt^2}x(t) = -4x(t)$. The characteristic equation is $m^2 = -4$ which has roots $m = \pm 2i$. From the table the solution is $x(t) = C_1 \cos 2t + C_2 \sin 2t$. If $x(0) = 0$ and $x'(0) = 1$ the resulting equations for the constants are $0 = C_1$ and $1 = 2C_2$. Thus the solution is $x(t) = \frac{1}{2}\sin 2t$.

Problem 7–2. The equation is $\frac{d^2}{dt^2}x(t) = -2\frac{d}{dt}x(t) - x(t)$. The characteristic equation is $m^2 = -2m - 1$ which has a double real root $m = -1$. The general solution is $x(t) = e^{-t}(C_1 + C_2 t)$. Since $x(0) = 1$ and $x'(0) = 0$, $1 = C_1$ and $0 = C_2 - C_1$. The solution is $x(t) = e^{-t}(1 + t)$.

Problem 7–3. False. The equation is not homogeneous.

Problem 7–4. The characteristic equation is $m^2 = 0$ which has $m = 0$ as a double real root. The solution is $x(t) = C_1 + C_2 t$. The equation can also be solved by integrating twice.

Problem 7–5. By separation of variables $\ln |A(t)| = t^3/3 + C$, from which the initial condition gives $C = 0$ and allows removal of the absolute value signs to give $A(t) = e^{t^3/3}$.

Problem 7–6. The characteristic equation is $m^2 + 4 = 0$ which has roots $m = \pm 2i$. The general solution is therefore $T(x) = C_1 \cos 2x + C_2 \sin 2x$. Using the boundary conditions shows that $C_1 = C_2 = 0$. Hence $T(x) = 0$ for all $0 \le x \le 2$.

Problem 7–7. The characteristic equation is $m^2 + V = 0$. There are 3 cases to consider: $V > 0$, $V = 0$, and $V < 0$. If $V > 0$ the roots are $m = \pm i\sqrt{V}$ and the general solution is $T(x) = C_1 \cos x\sqrt{V} + C_2 \sin x\sqrt{V}$. The boundary conditions give $C_1 = 0$ and $C_2 \sin 2\sqrt{V} = 0$. If $\sqrt{V}$ is a multiple of $\pi/2$ then C_2 can be non-zero. Write $\sqrt{V} = k\pi/2$, say. The general solution for such a V is $T(x) = C_2 \sin(k\pi x/2)$. If $V = 0$ the general solution is $T(x) = C_1 + C_2 x$ and the boundary conditions imply $C_1 = C_2 = 0$ and thus $T(x) = 0$ for all x. If $V < 0$ the general solution is $T(x) = C_1 e^{x\sqrt{-V}} + C_2 e^{-x\sqrt{-V}}$ and the boundary conditions again force $T(x) = 0$ for all x. Non-zero solutions are therefore possible only when $V = k^2\pi^2/4$ for some integer k, in which case the solution is $T(x) = C_2 \sin(k\pi x/2)$.

Problem 7–8. Integrating once gives $\frac{d}{dt}A(t) = 3t^2 + 8t + C$. A second integration gives $A(t) = t^3 + 4t^2 + Ct + D$. Using the initial conditions gives $D = 4$ and $C = 9$ so finally $A(t) = t^3 + 4t^2 + 9t + 4$.

Solutions to Exercises

Exercise 7–1. Simply plug this proposed solution into the equation.

Exercise 7–2. The mass returns to the equilibrium displacement of zero without any oscillation. The air resistance force is stronger than the spring.

Exercise 7–3. Simply plug the proposed solution back into the original equation.

Exercise 7–5. The displacement $x(t)$ is oscillatory.

Exercise 7–6. Plug the proposed solutions into the equation, keeping in mind that $i^2 = -1$.

Exercise 7–7. The initial conditions $x(0) = 1$ and $x'(0) = 0$ give the equations $C_2 = 1$ and $C_1 - 2C_2 = 0$. Hence $C_1 = 2$ and the solution is $x(t) = 2te^{-2t} + e^{-2t}$.

Exercise 7–8. Oscillatory solutions are obtained only when there are complex roots. Otherwise, the solution exhibits exponential decay only.

Exercise 7–9. Hint: What is the addition formula for cosine?

§8. Non-homogeneous Second Order Linear Equations

The undetermined coefficients method of solving the general non-homogeneous second order linear equation with constant coefficients is presented.

Example 8–1. Suppose a machine is attached to a 1 kilogram mass and the machine exerts a force of $\sin t$ newtons on the mass at time t. In addition the mass is attached to a spring having spring constant 4. The mass slides along a frictionless horizontal surface. The equation of motion is

$$\frac{d^2}{dt^2}x(t) = -4x(t) + \sin t$$

for $t > 0$. Here $x(t)$ is the displacement from the equilibrium position (which is taken to be 0) at time t.

The equation above is a second order linear equation with constant coefficients, but is not homogeneous since the function which takes the value 0 for all t is not a solution of the equation. The method of solution of such an equation is as follows.

(1) First find the general solution of the corresponding homogeneous equation. In this case, the corresponding homogeneous equation is $\frac{d^2}{dt^2}x(t) = -4x(t)$. Call this solution $h(t)$.

(2) Find (by some means) *some* solution of the non-homogeneous equation. Call this solution $p(t)$ a **particular solution**.

(3) The general solution of the original non-homogeneous equation is then $h(t) + p(t)$.

The method of solving the homogeneous equation has already been discussed. Here a method of finding a solution of the non-homogeneous equation will be presented.

Exercise 8–1. Find the general solution of $\frac{d^2}{dt^2}x(t) = -4x(t)$.

The **method of undetermined coefficients** is based on the fact that certain functions have derivatives which are again of the same form. For example, the derivative of any polynomial is again a polynomial (of lesser degree). In this particular case the trigonometric functions, sine and cosine, are essentially the derivatives of each other.

Example 8–2. To find *one* solution of the non-homogeneous equation above the method of undetermined coefficients suggests a trial solution of the form $A\cos t + B\sin t$ where A and B are as yet undetermined constants. Plugging this trial solution into the equation gives

$$-A\cos t - B\sin t = -4A\cos t - 4B\sin t + \sin t.$$

This equation will hold for all $t > 0$ only if $A = 0$ and $B = 1/3$. Thus $(1/3)\sin t$ is a solution of the non-homogenous equation. Using the general solution of the corresponding homogeneous equation then shows that $x(t) = C_1 \cos 2t + C_2 \sin 2t + (1/3)\sin t$ is the general solution of the non-homogeneous equation. The constants C_1 and C_2 would now be determined from the initial conditions in the usual way.

Exercise 8–2. Find C_1 and C_2 if the initial displacement is 1 and the initial velocity is 0.

If the initial guess is already a solution of the corresponding homogeneous equation, the guess should be multiplied by the independent variable (here t) in order to obtain a modified guess.

The method of undetermined coefficients is not guaranteed to work, and depends on the ingenuity of the practitioner to select the correct form of the trial solution.

The method of undetermined coefficients can also be used to solve non-homogeneous first order linear equations with constant coefficients.

Problems

Problem 8–1. Suppose in the example above the applied force is e^{-t} so that the equation of motion becomes $\frac{d^2}{dt^2}x(t) = -4x(t) + e^{-t}$. What is the solution of the equation if the initial displacement is 1 and the initial velocity is 0?

Problem 8–2. Find the general solution of the equation $A''(t) + A(t) = e^t$.

Problem 8–3. Find the general solution of the equation $A''(t) + A(t) = \sin t$.

Problem 8–4. Solve the equation $\frac{d^2}{dt^2}x(t)+9x(t) = 5$ with initial conditions $x(0) = 1$ and $x'(0) = 0$.

Problem 8–5. Solve the equation $\frac{d^2}{dt^2}x(t) + 9x(t) = t^2 + t$ with initial conditions $x(0) = 1$ and $x'(0) = 0$.

Problem 8–6. Solve the equation $\frac{d}{dt}x(t) = 4x(t) + t$ using the method of undetermined coefficients.

Problem 8–7. Solve $\frac{d^2}{dt^2}x(t) = -4x(t) + \sin 2t$. Hint: For the undetermined coefficients method try $At\cos 2t + Bt\sin 2t$.

Problem 8–8. True or False: There is a solution $B(t)$ of the equation $B''(t) + B'(t) + B(t) = 3$ for which $\lim_{t\to\infty} B(t) = \infty$.

Solutions to Problems

Problem 8–1. As before, the general solution to the homogeneous equation is $x(t) = C_1 \cos 2t + C_2 \sin 2t$. Since the exponential function is essentially its own derivative, the trial solution is Ae^{-t}. Substitution gives $Ae^{-t} = -4Ae^{-t} + e^{-t}$ from which $A = 1/5$. Hence the general solution to the equation is $x(t) = C_1 \cos 2t + C_2 \sin 2t + e^{-t}/5$. Using the initial conditions gives $1 = C_1 + 1/5$ and $0 = 2C_2 - 1/5$ so finally $x(t) = (4/5)\cos 2t + (1/10)\sin 2t + e^{-t}/5$.

Problem 8–2. Trying Me^t as a solution of the non-homogenous equation gives $2Me^t = e^t$, so $M = 1/2$. The general solution is $A(t) = A\cos t + B\sin t + 1/2e^t$.

Problem 8–3. Trying $Mt\cos t$ gives $M = -1/2$. The general solution is $A(t) = A\cos t + B\sin t - (1/2)t\cos t$.

Problem 8–4. The solution of the homogeneous equation is $x(t) = C_1 \cos 3t + C_2 \sin 3t$. A trial solution for the non-homogeneous equation is A, which gives $9A = 5$ or $A = 5/9$. The general solution of the non-homogeneous equation is therefore $x(t) = C_1 \cos 3t + C_2 \sin 3t + 5/9$. Using the initial conditions gives $1 = C_1 + 5/9$ and $0 = 3C_2$, so the solution is $x(t) = (4/9)\cos 3t + 5/9$.

Problem 8–5. The solution of the homogeneous equation is $x(t) = C_1 \cos 3t + C_2 \sin 3t$. A trial solution for the non-homogeneous equation is $At^2 + Bt + C$, which gives $2A + 9At^2 + 9Bt + 9C = t^2 + t$. Since this must hold for all t, $9A = 1$, $9B = 1$ and $2A + 9C = 0$. Solving these 3 equations gives $A = 1/9$, $B = 1/9$, and $C = -2/81$. The general solution of the non-homogeneous equation is therefore $x(t) = C_1 \cos 3t + C_2 \sin 3t + (1/9)t^2 + (1/9)t - 2/81$. Using the initial conditions gives $1 = C_1 - 2/81$ and $0 = 3C_2 + 1/9$, so the solution is $x(t) = (83/81)\cos 3t - (1/27)\sin 3t + (1/9)t^2 + (1/9)t - 2/81$.

Problem 8–6. The solution of the corresponding homogeneous equation $\frac{d}{dt}x(t) = 4x(t)$ is Ce^{4t}. The method of undetermined coefficients suggests $At + B$ as a trial solution. Plugging this into the equation gives $A = 4At + 4B + t$, and this holds for all t only if $A = -1/4$ and $B = -1/16$. The general solution of the equation is therefore $x(t) = Ce^{-4t} - \frac{1}{4}t - \frac{1}{16}$.

Problem 8–7. The solution of the homogeneous equation is $C_1 \cos 2t + C_2 \sin 2t$. The usual trial solution of $A\cos 2t + B\sin 2t$ will not work since it is a solution of the homogeneous equation. Using the suggested trial solution gives $A = -1/4$ and $B = 0$ so that $x(t) = C_1 \cos 2t + C_2 \sin 2t - \frac{1}{4}t\cos 2t$ is the general solution.

Problem 8–8. The general solution of the homogenous equation has limit 0, and $B(t) = 3$ solves the non-homogeneous equation. False.

Solutions to Exercises

Exercise 8–1. This is a homogeneous equation and the roots of the characteristic equation are $\pm 2i$. The general solution is therefore $x(t) = C_1 \cos 2t + C_2 \sin 2t$.

Exercise 8–2. The equations for the coefficients are $1 = C_1$ and $0 = 2C_2 + 1/3$.

§9. Additional Applications

Some additional applications of second order linear equations with constant coefficients are presented.

Example 9–1. An ideal pendulum consists of a point mass suspended from a fixed pivot point by a massless yet perfectly rigid wire of length L. Suppose the mass is a 2 kilogram mass and denote by $\theta(t)$ the angle between the suspending wire and the vertical direction at time t. In the absence of friction and air resistance the only force exerted on the mass is gravity. Considering the tangential component of the gravitational force and using Newton's Law gives

$$2L\frac{d^2}{dt^2}\theta(t) = -2g\sin\theta(t)$$

as the differential equation governing $\theta(t)$. This is a non-linear equation! If $\theta(t)$ is always small, the approximation $\sin\theta \approx \theta$ can be used to obtain an *approximately* valid equation

$$2L\frac{d^2}{dt^2}\theta(t) = -2g\theta(t)$$

which is linear and can be easily solved. This last equation describes the approximate motion of the pendulum.

Exercise 9–1. Solve the (approximate) equation with the initial conditions $\theta(0) = 0.1$ and $\theta'(0) = 0$.

Simple electrical circuits have as basic components resistors, capacitors, inductors, and voltage sources. A fundamental quantity in the analysis of a circuit is the charge $q(t)$ in the circuit at time t. The charge is measured in units called coulombs. The current $i(t)$ in the circuit at time t is defined to be $i(t) = \frac{d}{dt}q(t)$ and is measured in units called amperes. The change in electrical potential between two points in the circuit is measured in units called volts. The basic components in a circuit cause voltage drops (or rises, in the case of a voltage source) in different ways. According to Ohm's Law, a resistor causes a voltage drop which is proportional to the current; the proportionality constant is the resistance of the resistor and is measured in units called ohms. The voltage drop caused by a capacitor is proportional to the charge; the reciprocal of the proportionality constant is called the capacitance of the capacitor and is measured in units called farads. The voltage drop caused by an inductor is proportional to $\frac{d}{dt}i(t)$; the proportionality constant is called the inductance of the inductor and is measured in units called henrys.

Item	Notation/Properties
Charge	$q(t)$
Current	$i(t) = \frac{d}{dt}q(t)$
Resistor	Voltage Drop is $Ri(t)$ R is the resistance in ohms
Capacitor	Voltage Drop is $\frac{1}{C}q(t)$ C is the capacitance in farads
Inductor	Voltage Drop is $L\frac{d}{dt}i(t)$ L is the inductance in henrys

One fundamental fact about circuits is Kirchoff's Law which states that the sum of the voltage drops in any circuit loop must be 0. This is nothing more than the conservation of energy principle. Kirchoff's Law can be used to find the current or charge in simple circuits.

Example 9–2. A circuit loop contains a 5 volt battery, a 2 ohm resistor and a 1 farad capacitor. To find the charge in the circuit, apply Kirchoff's Law treating the voltage drops as -5, $2i(t)$ and $q(t)$ respectively to obtain

$$-5 + 2i(t) + q(t) = 0.$$

Using the definition of current $i(t) = \frac{d}{dt}q(t)$ then gives

$$2\frac{d}{dt}q(t) + q(t) - 5 = 0.$$

This equation can be easily solved. A reasonable initial condition here is that $q(0) = 0$.

Problems

Problem 9–1. You sit in a swing suspended by ropes 2 meters long. Your initial displacement from vertical is $\pi/12$ radians and you give yourself an initial boost so that your angular velocity is 0.5 radians/sec. After the initial boost you make no additional effort to influence the swinging. What is the maximum angular displacement from the vertical? Neglect air resistance and friction.

Problem 9–2. What happens in the previous problem if air resistance is equal in magnitude to velocity?

Problem 9–3. A circuit contains a 1 farad capacitor and a 1 henry inductor. Find the frequency of oscillation of the charge in the circuit.

Problem 9–4. A circuit contains a 1 henry inductor. What is the capacitance of the capacitor that should be added to obtain a circuit in which the charge oscillates with a frequency of 1000 hertz?

Problem 9–5. A circuit contains a 1 henry inductor, a 2000 ohm resistor, and a 1 microfarad capacitor. Initially there is no charge in the circuit and the initial current is 1 ampere. Find the charge in the circuit as a function of time.

Problem 9–6. What happens in the previous problem if the resistance of the resistor is really 1990 ohms? Is 2010 ohms?

Problem 9–7. A circuit contains a 1 farad capacitor, 1 henry inductor, and an AC generator whose voltage at time t is $\sin t$. Find the frequency of oscillation of the charge in the circuit. Assume that $q(0) = 0$ and $q'(0) = 1$.

Solutions to Problems

Problem 9–1. Let m denote your mass. Since the displacement is small, the differential equation for $\theta(t)$ here is $2m\frac{d^2}{dt^2}\theta(t) = -mg\theta(t)$, approximately. The solution is $\theta(t) = C_1\cos(t\sqrt{g/2} + C_2)$ in phase amplitude form. The initial conditions give $\pi/12 = C_1\cos C_2$ and $0.5 = -C_1\sqrt{g/2}\sin C_2$. Solving gives $C_1 = 0.345$ and $C_2 = -0.711$. The maximum angular displacement is therefore 0.345 radians or about 19.8 degrees.

Problem 9–2. The differential equation becomes $2m\frac{d^2}{dt^2}\theta(t) = -mg\theta(t) - 2\frac{d}{dt}\theta(t)$ or, upon division, $\frac{d^2}{dt^2}\theta(t) = -g/2\theta(t) - \frac{1}{m}\frac{d}{dt}\theta(t)$. Since m is at least 30 (for you), the solution is approximately the same as that in the previous problem. The maximum angular displacement is also about the same. This can be confirmed by solving for $\theta(t)$, and then finding the value of t that maximizes $\theta(t)$.

Problem 9–3. The equation for the charge is $\frac{d^2}{dt^2}q(t)+q(t) = 0$ which has general solution $q(t) = C_1\cos t + C_2\sin t$. The frequency of oscillation is therefore 1.

Problem 9–4. The equation for the charge is $\frac{d^2}{dt^2}q(t) + \frac{1}{C}q(t) = 0$ which has general solution $q(t) = C_1\cos t/\sqrt{C} + C_2\sin t/\sqrt{C}$. The frequency of oscillation is therefore $1/\sqrt{C}$. Hence $C = 0.000001$ farad, or 1 microfarad, does the trick.

Problem 9–5. The equation governing the charge is $\frac{d^2}{dt^2}q(t) + 2000\frac{d}{dt}q(t) + 1000000q(t) = 0$. The characteristic equation has a double real root at -1000, so the solution is $q(t) = e^{-1000t}(C_1 + C_2t)$. Since $q(0) = 0$ and $q'(0) = 1$, $0 = C_1$ and $1 = C_2$. Hence $q(t) = te^{-1000t}$.

Problem 9–6. If the resistance is 1990 ohms, the characteristic equation has conjugate complex roots $-995 \pm i5\sqrt{399}$. This leads to a periodic solution with exponentially decaying amplitude. If the resistance is 2010 ohms, the characteristic equation has distinct real roots $-1005 \pm 5\sqrt{401}$. This again leads to a rapidly decaying charge.

Problem 9–7. The differential equation is $\frac{d^2}{dt^2}q(t) + q(t) = \sin t$. The general solution of the homogeneous equation is $q(t) = C_1\cos t + C_2\sin t$. The trial solution for the undetermined coefficients method is then $At\cos t + Bt\sin t$. This leads to a particular solution of $-\frac{1}{2}t\cos t$ and the general solution of the non-homogeneous equation as $q(t) = -\frac{1}{2}t\cos t + C_1\cos t + C_2\sin t$. Using the condition $q(0) = 0$ gives $0 = C_1$. The condition $q'(0) = 1$ gives $1 = -(1/2) + C_2$. So finally $q(t) = -\frac{1}{2}t\cos t + \cos t + (3/2)\sin t$. The frequency of oscillation is 1.

Solutions to Exercises

Exercise 9–1. The characteristic equation of the homogeneous equation has roots $\pm i\sqrt{g/L}$.

§10. Systems of First Order Equations

Systems of first order differential equations arise in many contexts. The qualitative methods developed earlier can be used to understand some of the basic properties of the solutions of a system.

The first example illustrates a situation which gives rise to a system (pair) of differential equations.

Example 10–1. The classical Lotka-Volterra predator-prey model can be described as follows. An ecosystem contains 2 different species A and B. Species A is a predator that feeds on species B and only on species B. Species B depends only on the enviroment for nutrition and nutrition is unlimited. Denote by $A(t)$ and $B(t)$ the population sizes for the 2 species at time t. A possible model is then provided by the system of differential equations

$$\frac{d}{dt}A(t) = k_A A(t) + eA(t)B(t)$$
$$\frac{d}{dt}B(t) = fA(t)B(t) + k_B B(t).$$

The constants k_A and k_B represent the growth rates of the individual species populations in the absence of the other species. Thus $k_A < 0$ and $k_B > 0$ here. The constants e and f reflect the interdependency of the 2 species. The product $A(t)B(t)$ measures the chances of interaction between members of the two species. In the present context, the constant e should be positive while f should be negative.

Exercise 10–1. Why is $e > 0$ reasonable? Why is $f < 0$ reasonable? Why should $k_A < 0$ and $k_B > 0$ hold?

The system can be expressed in vector form as

$$\frac{d}{dt}(A(t), B(t)) = (k_A A(t) + eA(t)B(t), fA(t)B(t) + k_B B(t))$$

which is reminicent of equations studied earlier.

The Lotka-Volterra system is an example of an **autonomous system** since the independent variable t appears only through the unknown functions $A(t)$ and $B(t)$.

Qualitative information about the solution of the system can be obtained by methods similar to those used earlier.

Example 10–2. Suppose a Lotka-Volterra model for the predator-prey population is given by

$$\frac{d}{dt}(A(t), B(t)) = (-0.10A(t) + 0.01A(t)B(t), -0.01A(t)B(t) + 0.03B(t))$$

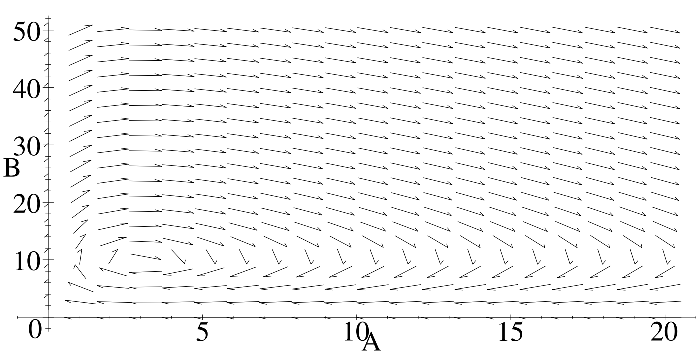
with the initial population $A(0) = 5$ and $B(0) = 30$. What is the population dynamics?

In parallel with the analysis in the one dimensional case, the first step in the analysis is to find the equilibrium solutions. This is done by setting the derivative vector equal to the zero vector and solving for $A(t)$ and $B(t)$. In this case, the two equilibrium solutions are $A(t) = B(t) = 0$ for all t, and $A(t) = 3$ and $B(t) = 10$ for all t.

Exercise 10–2. Verify that these are the only two equilibrium solutions.

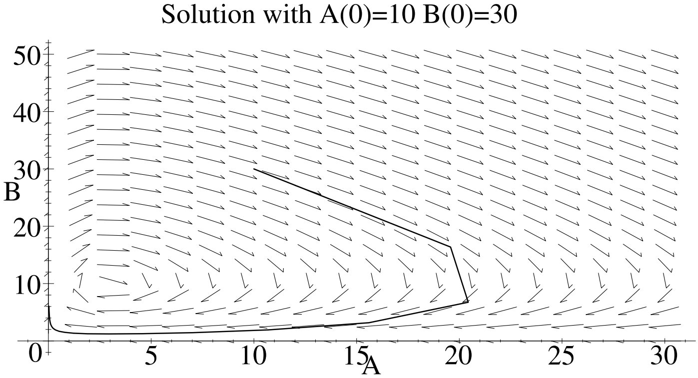
The second step is to plot the direction field. Since the system is autonomous, the derivative vector depends only on the values of the unknown functions. The direction field is then obtained by plotting a vector of length one proportional to $\frac{d}{dt}(A(t), B(t))$ at each point in the A-B plane.

Direction Field for Lotka-Volterra Model



As before, once the direction field is drawn, the behavior of a solution starting from any given initial conditions can be sketched by following the arrows.

Solution with A(0)=10 B(0)=30



Example 10–3. The Maple command to draw the direction field for the last example is

```
DEplot({diff(A(t), t) = −0.1 * A(t) + 0.01 * A(t) * B(t),
diff(B(t), t) = −.01 * A(t) * B(t) + 0.03 * B(t)}, [A(t), B(t)], 0..100, A = 0..30, B = 0..50);
```

and a solution starting with specified initial conditions is obtained with

```
DEplot({diff(A(t),t) = -0.1 * A(t) + 0.01 * A(t) * B(t),
diff(B(t),t) = -x.01 * A(t) * B(t) + 0.03 * B(t)},
[A(t),B(t)], 0..100, [[A(0) = 10, B(0) = 30]], A = 0..30, B = 0..50);.
```

Problems

Problem 10–1. Two countries, A and B, are suspicious of each other and respond by building up armaments. Suppose $A(t)$ and $B(t)$ are the arms budgets of the two countries at time t. Find a simple model that describes the growth of the arms budgets through time.

Problem 10–2. For the system

$$\frac{d}{dt}A(t) = -0.2A(t) + 0.9B(t) + 10$$
$$\frac{d}{dt}B(t) = 0.9A(t) - 0.2B(t) + 20$$

find the equilibrium solution. Is the equilibrium solution stable or unstable? Justify your answer.

Problem 10–3. Suppose that each of the two species in the example would follow logistic growth if the other species were absent. What would the system be in this case?

Solutions to Problems

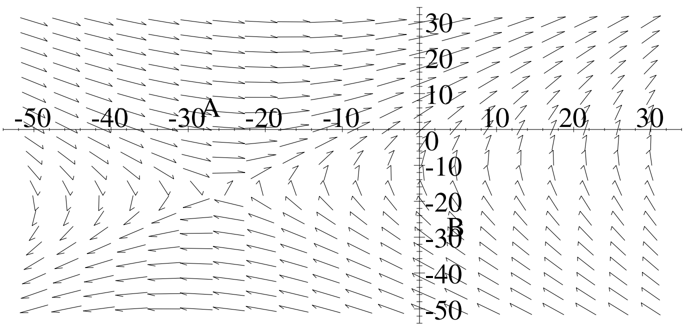
Problem 10–1. The basic rationale would be as follows. The rate of growth of the arms budget of country A is proportional to the arms budget of country B since the size of B's budget represents a threat. However, arms cost money, so this rate is also proportional to A's arms budget. There is also a constant growth factor representing inflation. A similar reasoning applies to country B. A simple model with these characteristics could be

$$\frac{d}{dt}A(t) = -aA(t) + bB(t) + c$$
$$\frac{d}{dt}B(t) = dA(t) - fB(t) + g$$

where all of the constants are positive.

Problem 10–2. The equilibrium solution occurs when $A = -25.97$ and $B = -16.88$. This solution is unstable, as is seen from the direction field plot: a small disturbance away from the equilibrium will cause the solution to move off to infinity or minus infinity.

Problem 10-2



Problem 10–3.

$$\frac{d}{dt}A(t) = k_A A(t)(M_A - A(t)) + eA(t)B(t)$$
$$\frac{d}{dt}B(t) = fA(t)B(t)(M_B - B(t)) + k_B B(t).$$

Solutions to Exercises

Exercise 10–1. Since species A is the predator, the more interactions between predator and prey, the better for species A. This explains why $e > 0$. These interactions are bad for species B, so $f < 0$. On the other hand, the more predators there are, the harder it will be for each one to find sufficient food. Thus $k_A < 0$. The more prey there are, the easier it will be for each one to avoid capture. Hence $k_B > 0$.

Exercise 10–2. Just plug the proposed solutions into the system of equations.

§11. Second Order Equations as Systems

The connection between second order linear equations and first order systems is developed here. This leads to qualitative methods for second order equations which parallel those developed for first order equations.

A simple trick can be employed to transform a single second order linear equation into a system.

Example 11–1. The equation $\frac{d^2}{dt^2}x(t) - 5\frac{d}{dt}x(t) + 7x(t) = 0$ can be transformed into a system as follows. Notice that the second order differential equation gives a relationship between $x''(t)$ and the functions $x(t)$ and $x'(t)$. This suggests that the vector $(x(t), x'(t))$ can be used to define a system. By making use of the differential equation, the derivative of this vector can be computed as follows.

$$\frac{d}{dt}\left(x(t), x'(t)\right) = \left(x'(t), 5x'(t) - 7x(t)\right).$$

Thus the derivative of the vector $(x(t), x'(t))$ is expressed in terms of the vector itself.

Since a second order equation can be represented by a system, this same method can be used to study the qualitative behavior of the solutions of a second order equation. In this case the direction field plot for the system is called the **phase portrait** of the second order equation. The reason for the terminology switch is that the direction field for the vector $(x(t), x'(t))$ gives information about the behavior of this vector, not just of the single unknown solution $x(t)$.

Problems

Problem 11–1. Write the second order equation $\frac{d^2}{dt^2}x(t) = -4x(t)$ as a system in vector form.

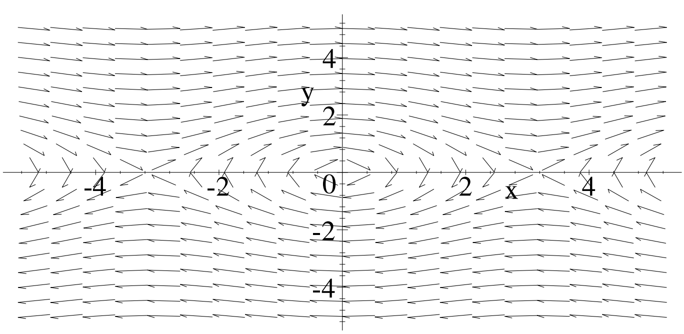
Problem 11–2. Write the second order equation $\frac{d^2}{dt^2}x(t) = 4x(t)$ as a system in vector form.

Problem 11–3. Which of the direction fields below is the direction field for the system derived from the second order equation $\frac{d^2}{dt^2}x(t) = -4x(t)$? Use the direction field corresponding to this equation to sketch the solution satisfying $x(0) = 1$ and $x'(0) = 0$.

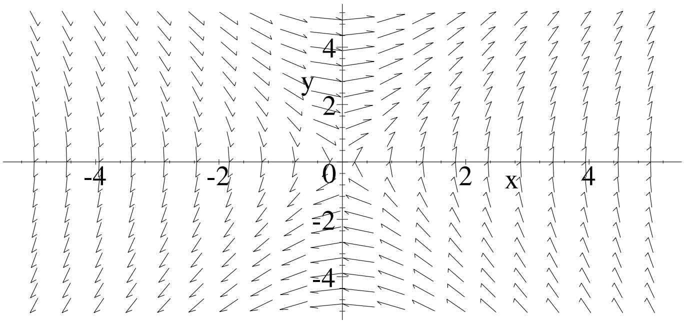
Problem 11–4. Which of the direction fields below is the direction field for the system derived from the second order equation $\frac{d^2}{dt^2}x(t) = 4x(t)$? Use the direction field corresponding to this equation to sketch the solution satisfying $x(0) = 1$ and $x'(0) = 0$.

Problem 11–5. Which of the direction fields below is the direction field for the system derived from the second order equation $\frac{d^2}{dt^2}x(t) = -\sin(x(t))$? Use the direction field corresponding to this equation to sketch the solution satisfying $x(0) = 1$ and $x'(0) = 0$.

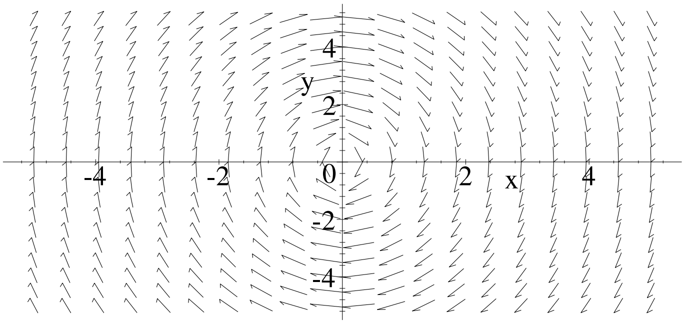
Direction Field A



Direction Field B



Direction Field C



Solutions to Problems

Problem 11–1. Using the equation gives

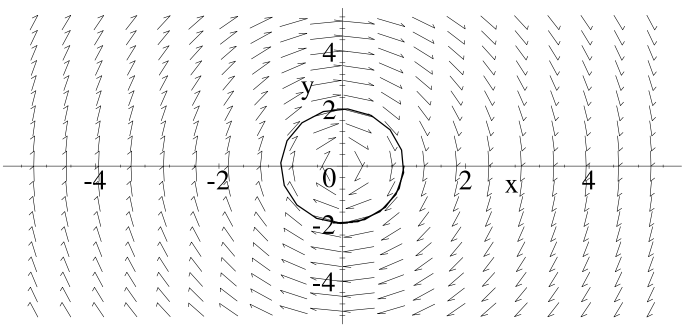
$$\frac{d}{dt}\left(x(t), x'(t)\right) = \left(x'(t), -4x(t)\right).$$

Problem 11–2. Using the equation gives

$$\frac{d}{dt}\left(x(t), x'(t)\right) = \left(x'(t), 4x(t)\right).$$

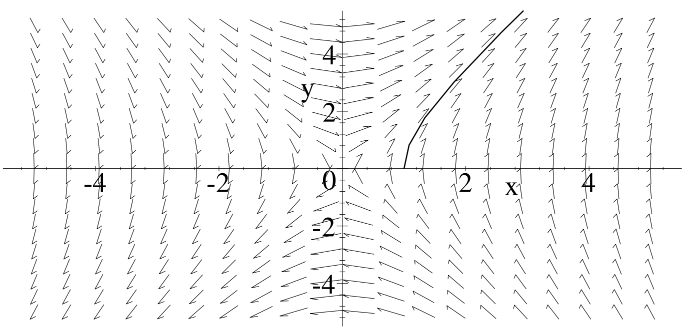
Problem 11–3.

Solution of x''(t)= -4x(t) with x(0)=1 and x'(0)=0



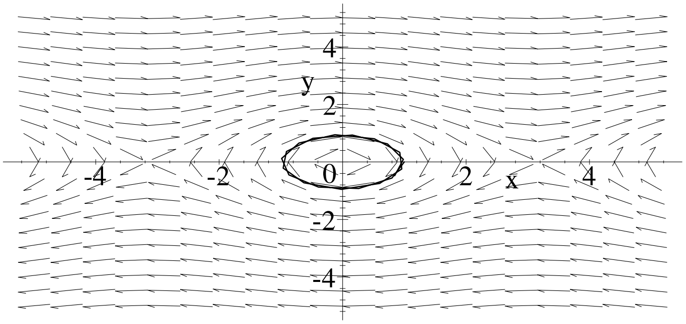
Problem 11–4.

Solution of x''(t)= 4x(t) with x(0)=1 and x'(0)=0



Problem 11–5.

Solution of x''(t)= -sin(x(t)) with x(0)=1 and x'(0)=0



§12. The Laplace Transform Method

An alternate means of solving linear differential equations with constant coefficients is developed which replaces many calculus operations with algebraic ones. The method also allows the solution of some equations which would be difficult to handle by conventional means.

In order for calculus operations to be replaced by algebraic ones, the calculus operations are actually done in the background. The means by which this is accomplished is as follows. Given a function $f(t)$, define a new function $\mathcal{L}\{f(t)\}(s)$ of the variable s by the formula

$$\mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

This new function $\mathcal{L}\{f(t)\}(s)$ is called the **Laplace transform** of the function $f(t)$.

Example 12–1. The Laplace transform of the constant function 1 is easily computed to be $\mathcal{L}\{1\}(s) = \int_0^\infty e^{-st}\, dt = 1/s$.

Example 12–2. Using integration by parts it is easy to compute the Laplace transform of t.

Exercise 12–1. Use integration by parts to compute the Laplace transform of t.

Since the ultimate objective is to solve differential equations using Laplace transforms, the relationship between the Laplace transform of a derivative and the Laplace transform of the original function will play an important role. Again, integration by parts provides the computational method.

$$\begin{aligned}\mathcal{L}\{f(t)\}(s) &= \int_0^\infty e^{-st} f(t)\, dt \\ &= -f(t)e^{-st}/s\Big|_{t=0}^{t=\infty} + (1/s)\int_0^\infty e^{-st} f'(t)\, dt \\ &= f(0)/s + (1/s)\mathcal{L}\{f'(t)\}(s).\end{aligned}$$

This formula is one of the central results in the Laplace transform theory.

Example 12–3. This formula provides an alternate way of computing $\mathcal{L}\{t\}(s)$. Using the formula, $\mathcal{L}\{t\}(s) = 0/s + (1/s)\mathcal{L}\{1\}(s) = (1/s)(1/s) = 1/s^2$. The integration by parts required when computing directly has been done once for all in the general formula!

Exercise 12–2. How is $\mathcal{L}\{f''(t)\}(s)$ related to $\mathcal{L}\{f(t)\}(s)$?

Example 12–4. Using the formula gives $\mathcal{L}\{e^{mt}\}(s) = (1/s) + (1/s)\mathcal{L}\{me^{mt}\}(s) = (1/s) + (m/s)\mathcal{L}\{e^{mt}\}(s)$. Solving this equation yields $\mathcal{L}\{e^{mt}\}(s) = 1/(s-m)$.

Example 12–5. How are Laplace transforms used to solve differential equations? The method is illustrated with the simple equation $A'(t) = A(t)$ with initial condition $A(0) = 1$. The first step is to compute the Laplace transform of both sides of the equation. Making use of the key identity above gives $s\mathcal{L}\{A(t)\}(s) - A(0) = \mathcal{L}\{A(t)\}(s)$. Using the initial condition this simplifies to $s\mathcal{L}\{A(t)\}(s)-1 = \mathcal{L}\{A(t)\}(s)$. The second step is to solve this algebraic equation for the Laplace transform of the unknown function. Here this gives $\mathcal{L}\{A(t)\}(s) = 1/(s-1)$. The final step is to identify this Laplace transform. Using the result of the previous exercise $\mathcal{L}\{e^t\}(s) = 1/(s-1)$. Hence $A(t) = e^t$.

The three steps in using Laplace transforms to solve differential equations are:

(1) Compute the Laplace transform of both sides of the differential equation. Use the key identity to express the Laplace transform of the derivatives of the unknown function in terms of the Laplace transform of the unknown function itself. Use the initial conditions as needed.

(2) Solve the resulting algebraic equation from the first step to get an algebraic expression for the Laplace transform of the unknown function.

(3) Write the expression for the Laplace transform in a recognizable form, and deduce the unknown function.

The last step is the one in which considerable skill may be required. Certainly a knowledge of the Laplace transform of commonly occurring functions is needed.

Example 12–6. The formula above can be used to compute the Laplace transform of $\cos t$ and $\sin t$. The formula gives $\mathcal{L}\{\cos t\}(s) = 1/s - (1/s)\mathcal{L}\{\sin t\}(s)$ and $\mathcal{L}\{\sin t\}(s) = (1/s)\mathcal{L}\{\cos t\}(s)$. These two equations in the two unknowns $\mathcal{L}\{\cos t\}(s)$ and $\mathcal{L}\{\sin t\}(s)$ can now be solved.

A second useful general formula is obtained by manipulation of the definition of the Laplace transform, as follows: $\mathcal{L}\{e^{bt}f(t)\}(s) = \int_0^\infty e^{-st}e^{bt}f(t)\,dt = \int_0^\infty e^{-(s-b)t}f(t)\,dt = \mathcal{L}\{f(t)\}(s-b)$.

Exercise 12–3. Use this last formula to find the Laplace transform of te^t.

Example 12–7. Euler's identity, $e^{imt} = \cos mt + i\sin mt$, can be used along with this second computational formula to obtain $\mathcal{L}\{t\sin mt\}(s)$ and $\mathcal{L}\{t\cos mt\}(s)$. On the one hand, $\mathcal{L}\{te^{imt}\}(s) = 1/(s-im)^2 = (s+im)^2/(s^2+m^2)^2 = ((s^2-m^2)+2ism)/(s^2+m^2)^2$ by the preceding exercise and properties of complex numbers. On the other hand, $\mathcal{L}\{te^{imt}\}(s) = \mathcal{L}\{t\cos mt\}(s) + i\mathcal{L}\{t\sin mt\}(s)$. Equating the real parts of these two expressions gives $\mathcal{L}\{t\cos mt\}(s) = (s^2 - m^2)/(s^2 + m^2)^2$.

Exercise 12–4. What is $\mathcal{L}\{t\sin mt\}(s)$?

The most useful facts about Laplace transforms are summarized in the following table.

Table of Laplace Transforms

$f(t)$	$\mathcal{L}\{f(t)\}(s)$
$f'(t)$	$s\mathcal{L}\{f(t)\}(s) - f(0)$
$f''(t)$	$s^2\mathcal{L}\{f(t)\}(s) - sf(0) - f'(0)$
$e^{bt}f(t)$	$\mathcal{L}\{f(t)\}(s-b)$
t^n	$n!/s^{n+1}$
e^{mt}	$1/(s-m)$
$\cos mt$	$s/(s^2+m^2)$
$\sin mt$	$m/(s^2+m^2)$
$t^n e^{mt}$	$n!/(s-m)^{n+1}$

All of the linear equations with constant coefficients from the earlier sections, homogeneous or not, can now be solved using Laplace transforms. Solution of non-homogeneous equations usually requires the use of partial fractions.

Example 12–8. To solve the equation $B''(t) + B(t) = e^t$ with the initial conditions $B(0) = 0$ and $B'(0) = 0$ using Laplace transforms the first step is to compute the Laplace transform of both sides of the equation. This gives $s^2\mathcal{L}\{B(t)\}(s) + \mathcal{L}\{B(t)\}(s) = 1/(s-1)$. Simple algebra yields $\mathcal{L}\{B(t)\}(s) = \dfrac{1}{(s^2+1)(s-1)}$. To recognize this fraction as a Laplace transform, use partial fractions to rewrite it as $\dfrac{1}{(s^2+1)(s-1)} = \dfrac{(-1/2)s-(1/2)}{s^2+1} + \dfrac{1/2}{s-1} = -(1/2)\dfrac{s}{s^2+1} - (1/2)\dfrac{1}{s^2+1} + (1/2)\dfrac{1}{s-1}$. The pieces are each recognizable from the table, so that $B(t) = -(1/2)\cos t - (1/2)\sin t + (1/2)e^t$.

Problems

Problem 12–1. Use the Laplace transform method to solve the equation $A''(t) + A(t) = 0$ with the initial condition $A(0) = 0$ and $A'(0) = 1$. Check your answer by solving the equation using the more familiar method.

Problem 12–2. Compute $\mathcal{L}\{\int_0^t f(x)\,dx\}(s)$ in terms of $\mathcal{L}\{f(t)\}(s)$.

Problem 12–3. Find the current $i(t)$ in a circuit containing a 1 ohm resistor, 1 farad capacitor, and a 6 volt battery in a single loop. Assume the initial charge in the circuit is $q(0)$.

Problem 12–4. Compute $\mathcal{L}\{e^{bt}\cos mt\}(s)$ and $\mathcal{L}\{e^{bt}\sin mt\}(s)$.

Problem 12–5. Use the Laplace transform method to solve the equation $C''(t) + C(t) = \sin t$ with the initial conditions $C(0) = 0$ and $C'(0) = 0$.

Problem 12–6. Choose some problems involving linear differential equations with constant coefficients from the preceding sections and solve them using Laplace transforms. Check your answer by comparing it to the solution obtained using the previous methods.

Solutions to Problems

Problem 12–1. Taking Laplace transforms gives $s^2\mathcal{L}\{A(t)\}(s)-1+\mathcal{L}\{A(t)\}(s) = 0$ so that $\mathcal{L}\{A(t)\}(s) = 1/(s^2+1)$. Hence $A(t) = \cos t$.

Problem 12–2. Using the fundamental computational formula and the Fundamental Theorem of Calculus gives $\mathcal{L}\{\int_0^t f(x)\,dx\}(s) = 0/s + (1/s)\mathcal{L}\{f(t)\}(s)$.

Problem 12–3. Kirchoff's law gives $i(t)+q(t) = 6$. Since $q(t) = q(0)+\int_0^t i(x)\,dx$, this becomes $i(t) + q(0) + \int_0^t i(x)\,dx = 6$. Taking Laplace transforms of both sides and using the previous problem then gives $\mathcal{L}\{i(t)\}(s) + (1/s)\mathcal{L}\{i(t)\}(s) = (6-q(0))/s$ from which $\mathcal{L}\{i(t)\}(s) = (6-q(0))/(s+1)$. Hence $i(t) = (6-q(0))e^{-t}$.

Problem 12–4. Using the second computational formula gives $\mathcal{L}\{e^{bt}\cos mt\}(s) = \mathcal{L}\{\cos mt\}(s-b) = (s-b)/((s-b)^2+m^2)$ and $\mathcal{L}\{e^{bt}\sin mt\}(s) = m/((s-b)^2+m^2)$.

Problem 12–5. Taking Laplace transforms gives $s^2\mathcal{L}\{C(t)\}(s) + \mathcal{L}\{C(t)\}(s) = 1/(s^2+1)$, which in turn becomes $\mathcal{L}\{C(t)\}(s) = 1/(s^2+1)^2$. Hence $C(t) = t\sin t$.

Solutions to Exercises

Exercise 12–1. $\mathcal{L}\{t\}(s) = \int_0^\infty te^{-st}\,dt = -te^{-st}/s\Big|_{t=0}^{t=\infty} + (1/s)\int_0^\infty e^{-st}\,dt = 1/s^2$.

Exercise 12–2. Apply the formula twice to obtain $\mathcal{L}\{f(t)\}(s) = f(0)/s + (1/s)\mathcal{L}\{f'(t)\}(s) = f(0)/s + (1/s)(f'(0)/s + (1/s)\mathcal{L}\{f''(t)\}(s)) = f(0)/s + f'(0)/s^2 + (1/s^2)\mathcal{L}\{f''(t)\}(s)$.

Exercise 12–3. By the formula, $\mathcal{L}\{te^t\}(s) = \mathcal{L}\{t\}(s-1) = 1/(s-1)^2$.

Exercise 12–4. Equating the imaginary parts of the two expressions in the preceding example gives $\mathcal{L}\{t\sin mt\}(s) = 2sm/(s^2+m^2)^2$.

§13. Input–Output Systems

An alternate way of interpreting a differential equation is developed here. As a consequence of this viewpoint, a new function is introduced and its properties are studied.

Especially in the engineering field, the operation of a machine is often viewed schematically as processing an input of some kind in order to produce an output. A basic engineering problem is to model the relationship between the input and output for a particular machine. If an adequate model for this relationship is known, the input required in order to obtain the desired output can be determined.

Example 13–1. As a simple example, consider an electric circuit. What is the relationship between the input voltage $I(t)$ at time t and the output voltage $O(t)$ at time t for the circuit? Suppose the input voltage, a one ohm resistor, and a one farad capacitor form a loop; the output voltage forms a loop with this same capacitor. Kirchoff's law applied to the input loop gives

$$i(t) + q(t) = I(t)$$

where $i(t)$ and $q(t)$ are the current and charge in the input loop. Kirchoff's law applied to the output loop gives

$$q(t) = O(t).$$

Since $i(t) = q'(t) = O'(t)$ from this second equation, substitution gives

$$O'(t) + O(t) = I(t)$$

as the relationship between the input and the output voltages.

Similar equations would result from other possible types of simple circuits. Thus one interpretation of the solution of a non-homogeneous linear differential equation is as the output of a system which is supplied with the non-homogeneous part of the equation as the input.

Example 13–2. What equation relates the input and output if the capacitor in the previous example is replaced by a one henry inductor?

This point of view makes certain simple types of functions candidates for the input function of the input–output system.

Example 13–3. Consider a circuit for which the input voltage is 1 volt for 1 time unit and then is switched off. The input function $I(t)$ is 1 for $0 \le t \le 1$ and 0 for all other values of t.

Example 13–4. The input voltage might be alternately 1 volt or -1 volt, each input lasting for 1 time unit.

These sorts of inputs are quite common in practice. Fortunately, such inputs can be described relatively simply in terms of a special building block function. The **Heaviside function**, denoted $H(t)$, is the function which takes the value 0 when $t < 0$ and the value 1 when $t \geq 0$.

Example 13–5. The input voltage which is 1 volt for 1 time unit is therefore described by the function $H(t) - H(t-1)$.

Example 13–6. The input voltage which is alternately 1 volt or -1 volt for 1 time unit is $\sum_{k=0}^{\infty}(-1)^k(H(t-k) - H(t-k-1))$.

In order to conveniently solve differential equations involving such functions, the Laplace transform method will be used. Direct computation gives

$$\begin{aligned}\mathcal{L}\{H(t-k)\}(s) &= \int_0^\infty e^{-st}H(t-k)\,dt \\ &= \int_k^\infty e^{-st}\,dt \\ &= e^{-sk}/s.\end{aligned}$$

The second equality comes from the fact that $H(t-k)$ is zero if $t < k$ and $H(t-k) = 1$ for $t \geq k$.

Exercise 13–1. Compute the Laplace transform of $\sum_{k=0}^{\infty}(-1)^k(H(t-k)-H(t-k-1))$.

As usual, when the Laplace transform method is used the essential difficulty is performing the last step in obtaining a solution. The following formula is extremely helpful in completing this step. For any $r > 0$,

$$\begin{aligned}\mathcal{L}\{f(t-r)H(t-r)\}(s) &= \int_0^\infty e^{-st}f(t-r)H(t-r)\,dt \\ &= \int_r^\infty e^{-st}f(t-r)\,dt \\ &= \int_0^\infty e^{-s(u+r)}f(u)\,du \\ &= e^{-sr}\mathcal{L}\{f(t)\}(s).\end{aligned}$$

Example 13–7. What function has Laplace transform e^{-s}/s? Since $\mathcal{L}\{H(t)\}(s) = 1/s$, $e^{-s}/s = e^{-s}\mathcal{L}\{H(t)\}(s) = \mathcal{L}\{H(t-1)\}(s)$. So the function is $H(t-1)$.

Example 13–8. What is the solution of the equation $A'(t)+A(t) = H(t)-H(t-1)$ with the initial conditions $A(0) = 0$ and $A'(0) = 0$? As usual, take Laplace transforms of both sides to obtain $s\mathcal{L}\{A(t)\}(s) + \mathcal{L}\{A(t)\}(s) = 1/s - e^{-s}/s$ which gives $\mathcal{L}\{A(t)\}(s) = \dfrac{1}{s(s+1)} + \dfrac{e^{-s}}{s(s+1)}$. Now by partial fractions $\dfrac{1}{s(s+1)} = \dfrac{1}{s} + \dfrac{-1}{s+1}$ and the first part of the solution is $1 - e^{-t}$. Since the second part of the Laplace transform is e^{-s} times the first part, the second part of the solution is $(1 - e^{-(t-1)})H(t-1)$. Hence $A(t) = 1 - e^{-t} + (1 - e^{-(t-1)})H(t-1)$.

Problems

Problem 13–1. True or False: If $H(t)$ is the Heaviside function and $f(t)$ is any function then $\mathcal{L}\{f(t)H(t)\}(s) = \mathcal{L}\{f(t)\}(s)$.

Problem 13–2. Define the function $G(t)$ by the formula $G(t) = \sum_{j=0}^{\infty} 2^j H(t-j)$. Here $H(t)$ is the Heaviside function. Sketch the graph of $G(t)$ for $-2 \leq t \leq 3$. Find the Laplace transform of $G(t)$, simplifying as much as possible. (Your final answer should not be a series.)

Problem 13–3. If $\mathcal{L}\{f(t)\}(s) = \dfrac{1}{s(1-e^{-s})}$, what is $f(t)$? Hint: Geometric series.

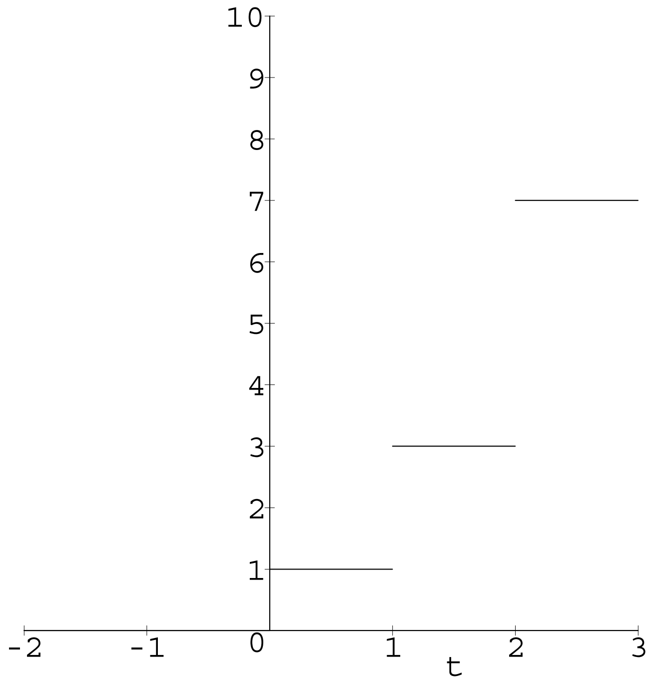
Problem 13–4. Solve the equation $A''(t) - A(t) = H(t) - H(t-1)$ under the initial conditions $A(0) = 0$ and $A'(0) = 0$. What is $\lim_{t\to\infty} A(t)$?

Problem 13–5. Refer to the previous problem. How would you solve the equation $A''(t) - A(t) = H(t) - H(t-1)$ under the initial conditions $A(0) = 1$ and $A'(0) = 5$?

Solutions to Problems

Problem 13–1. True, since the range of integration used to compute Laplace transforms is from zero to infinity, and on this interval the Heaviside function takes the value 1.

Problem 13–2.



Since $\mathcal{L}\{H(t-j)\}(s) = e^{-js}/s$, $\mathcal{L}\{G(t)\}(s) = \sum_{j=0}^{\infty} \mathcal{L}\{2^j H(t-j)\}(s) = \sum_{j=0}^{\infty} 2^j e^{-js}/s =$ $(1/s)\sum_{j=0}^{\infty}(2e^{-s})^j = \dfrac{1}{s(1-2e^{-s})}$, by summing the geometric series.

Problem 13–3. Using the geometric series formula gives $\dfrac{1}{s(1-e^{-s})} = \sum_{k=0}^{\infty} \dfrac{e^{-ks}}{s}$. Since $\mathcal{L}\{H(t-k)\}(s) = e^{-ks}/s$, this means that $f(t) = \sum_{k=0}^{\infty} H(t-k)$.

Problem 13–4. Taking Laplace transforms gives $(s^2-1)\mathcal{L}\{A(t)\}(s) = 1/s - e^{-s}/s$. Partial fractions shows that $\dfrac{1}{s(s^2-1)} = \dfrac{-1}{s} + \dfrac{1/2}{s-1} + \dfrac{1/2}{s+1}$. Hence, $A(t) = -1 + (1/2)e^t + (1/2)e^{-t} - (-1 + (1/2)e^{t-1} + (1/2)e^{1-t})H(t-1)$. The limit is infinity.

Problem 13–5. Find the solution of the homogeneous equation $A''(t) - A(t) = 0$ which satisfies $A(0) = 1$ and $A'(0) = 5$ and add it to the solution of the previous problem.

Solutions to Exercises

Exercise 13–0. The equations from applying Kirchoff's law to the two loops are $i(t) + i'(t) = I(t)$ and $i'(t) = O(t)$. Hence $O(t) + O(0) + \int_0^t O(x)\,dx = I(t)$.

Exercise 13–1. The Laplace transform is $\sum_{k=0}^{\infty}(-1)^k(e^{-ks} - e^{-(k+1)s})/s = (1 - e^{-s})/(s(1-e^{-s})$, by using the geometric series formula $\sum_{k=0}^{\infty} x^k = 1/(1-x)$.

§14. Some Theoretical Considerations

The concept of linearity is explored in the context of differential equations.

One of the more interesting developments in modern mathematics is the number of situations which have been identified in which concepts from linear algebra play a significant role. Here the interaction between linear algebra and differential equations will be briefly explored.

The phrase 'linear algebra' is associated with two central concepts: vectors and linear transformations. In order to make the connection between linear algebra and differential equations apparent, vectors and linear transformations must be identified in the differential equations setting.

The important properties of vectors are that the sum of two vectors is again a vector, and the product of a number (scalar) and a vector is again a vector. The key realization is that functions also have these two properties. The sum of two functions is again a function; the product of a number and a function is again a function. Functions can therefore be viewed as vectors (points) in some space.

A linear trasformation L is a function which maps vectors into vectors and obeys the linearity property $L(v + w) = L(v) + L(w)$. In this last equation read the phrase 'the derivative of' in place of L; differentiation can be viewed as a linear transformation on the newly found set of vectors (which used to be thought of as functions). The connection between differential equations and linear algebra is now almost complete.

Example 14–1. Define a linear transformation L on the space of functions of the variable t by the formula $L(f(t)) = \frac{d}{dt}f(t) - 7f(t)$. The kernel of L is then the set of functions $f(t)$ for which $L(f(t)) = 0$. Notice that the kernel of L is also the set of solutions of the differential equation $\frac{d}{dt}f(t) - 7f(t) = 0$. This sort of reasoning can be used to show that the solution set is one dimensional; this is why there is only one arbitrary constant in the solution of first order linear differential equations with constant coefficients.

Example 14–2. As a second example the **annihilator method** of solving second order linear non-homogeneous equations is developed. The method of undetermined coefficients can sometimes be difficult to apply. The annihilator method is often easier to use. Suppose the equation to be solved is

$$\frac{d^2}{dt^2}x(t) + 4x(t) = \sin 2t.$$

The general solution of the corresponding homogeneous equation is $C_1 \cos 2t + C_2 \sin 2t$. The method of undetermined coefficients will fail here if applied directly since both $\cos 2t$ and $\sin 2t$ solve the homogeneous equation. Define a linear transformation L on the space of functions by $L(f(t)) = \frac{d^2}{dt^2}f(t) + 4f(t)$. The equation being studied can then be written as

$$L(x(t)) = \sin 2t.$$

The annihilator method consists of two steps. First, find a linear transformation which maps $\sin 2t$ to zero. This is the same as finding a differential equation satisfied by $\sin 2t$. One choice of linear transformation is $A(f(t)) = \frac{d^2}{dt^2}f(t) + 4f(t)$ (that is, $A = L$). Second, find the general solution of the homogeneous equation $A(L(f(t))) = 0$. Since $A(L(f(t))) = \frac{d^4}{dt^4}f(t) + 8\frac{d^2}{dt^2}f(t) + 16f(t)$ the general solution of this equation can be found by extending the methods used earlier for second order linear homogeneous equations. In this case the general solution is $C_1 \cos 2t + C_2 \sin 2t + C_3 t \cos 2t + C_4 t \sin 2t$. This provides a trial solution of the non-homogeneous equation $L(x(t)) = \sin 2t$ which is guaranteed to work.

Exercise 14–1. Find C's that make this trial solution a solution of the original non-homogeneous equation.

Exercise 14–2. Find the general solution of the equation $L(x(t)) = \sin 2t$. Graph the solution which satisfies the initial conditions $x(0) = 0$ and $x'(0) = 1$.

This example illustrates the phenomenon of resonance. The driving force in the equation has the same frequency as the frequency of the solutions of the homogeneous equation. The reinforcing action of the driving force increases the energy of the system unboundedly.

Problems

Problem 14–1. Find the general solution of the equation $\frac{d^2}{dt^2}x(t) - 4x(t) = e^{2t}$ using the annihilator method.

Problem 14–2. Find the general solution of $\frac{d^2}{dt^2}x(t) - 4x(t) = t^2$ by the annihilator method.

Solutions to Problems

Problem 14–1. The function e^{2t} satisfies the differential equation $\frac{d}{dt}x(t) = 2x(t)$. The characteristic equation for the annihilator method is then $(m-2)(m^2-4) = 0$. The general soltution is then $C_1e^{-2t} + e^{2t}(C_2 + C_3t)$. The constant C_3 is then determined by substitution.

Problem 14–2. The function t^2 satisfies the differential equation $\frac{d^3}{dt^3}x(t) = 0$. The characteristic equation for the annihilator method is then $m^3(m^2-4) = 0$. The general solution is $C_1\cos 2t + C_2\sin 2t + C_3t^2 + C_4t + C_5$. The constants C_3, C_4, and C_5 are then determined by substitution.

Solutions to Exercises

Exercise 14–1. Hint: The constants C_1 and C_2 can be taken to be zero since $\cos 2t$ and $\sin 2t$ solve the homogeneous equation. Then $x''(t) + 4x(t) = -4C_3 \sin 2t + 4C_4 \cos 2t$. This is equal to $\sin 2t$ if $C_3 = -1/4$ and $C_4 = 0$. The general solution to the non-homogeneous equation is $x(t) = C_1 \cos 2t + C_2 \sin 2t - t/4 \cos 2t$.

Exercise 14–2. The equations for the coefficients are $0 = C_1$ and $1 = 2C_2 - 1/4$.