Convergence of Series of Independent Random Variables

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1 Introduction

Suppose $X_1, X_2, \ldots$ is a sequence of independent random variables defined on a common probability space. Denote by $S_n = \sum_{j=1}^{n} X_j$. The objective of this note is to show that if S_n converges in distribution then S_n converges almost surely.

The converse of this result is easily proved using characteristic functions and the Dominated Convergence Theorem. The present result seems to be much less well known.

For the proof of the result, three basic facts will be required.

2 The Convergence Set

The first fact actually applies to any sequence of random variables. Denote by C the possibly empty set on which the sequence $\{S_n\}$ converges. The Cauchy criterion shows that

$$C = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m+1}^{\infty} [|S_n - S_m| \leq 1/k].$$

In order to make all of the sequences of sets nested, note that

$$\begin{aligned} \bigcap_{n=m+1}^{\infty} [|S_n - S_m| \leq 1/k] &= \bigcap_{n=m+1}^{\infty} \bigcap_{r=m+1}^{n} [|S_r - S_m| \leq 1/k] \\ &= \bigcap_{n=m+1}^{\infty} [\max_{m<r\leq n} |S_r - S_m| \leq 1/k] \end{aligned}$$

so that

$$C = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m+1}^{\infty} [\max_{m<r\leq n} |S_r - S_m| \leq 1/k].$$

Now because of the nesting property of the sets and the Monotone Convergence Theorem,

$$P[C] = \lim_{k\to\infty} \lim_{m\to\infty} \lim_{n\to\infty} P[\max_{m<r\leq n} |S_r - S_m| \leq 1/k]$$

as an iterated limit.

3 Ottaviani's Inequality

The following inequality will be used to estimate the probability occurring in the convergence criteria. This result has been attributed to Ottaviani. The inequality is that for any $n > m$ and any $\varepsilon > 0$

$$P[\max_{m<r\leq n} |S_r - S_m| > 2\varepsilon] \cdot \min_{m<r\leq n} P[|S_n - S_r| \leq \varepsilon] \leq P[|S_n - S_m| > \varepsilon].$$

To establish the inequality, let $n > m$ and consider the event $[|S_n - S_m| > \varepsilon]$. Among the possible ways this event can occur are the ones at which there is a k

where for the first time $|S_k - S_m| > 2\varepsilon$ and then $|S_n - S_k| \leq \varepsilon$. Clearly these types of possibilities do not include all the ways the event can occur. Thus

$$\bigcup_{k=m+1}^{n} [\max_{m<r\leq k-1} |S_r - S_m| \leq 2\varepsilon, |S_k - S_m| > 2\varepsilon, |S_n - S_k| \leq \varepsilon] \subset [|S_n - S_m| > \varepsilon]$$

and the sets in the union are disjoint. Computing probabilities and using independence gives

$$\sum_{k=m+1}^{n} P[\max_{m<r\leq k-1} |S_r - S_m| \leq 2\varepsilon, |S_k - S_m| > 2\varepsilon] \cdot P[|S_n - S_k| \leq \varepsilon]$$
$$\leq P[|S_n - S_m| > \varepsilon].$$

The result now follows by replacing the second factor in the sum by the minimum expression of the conclusion (which only makes the sum smaller), and then realizing the sum of the remaining terms is exactly the first probability in the conclusion.

4 A Fact About Characteristic Functions

For a random variable Y, denote by ϕ_Y the characteristic function of Y. Suppose that for some $t \neq 0$ $\phi_Y(t) = 1$. Then $1 = E[\exp\{itY\}] = E[\cos(tY)] + iE[\sin(tY)]$, so that $E[1 - \cos(tY)] = 0$. Since $1 - \cos(tY) \geq 0$, this implies that $P[tY \in 2\pi Z] = 1$, where Z is the set of integers.

If there are two non-zero values s and t with $\phi_Y(s) = \phi_Y(t) = 1$, then $P[Y \in (2\pi/t)Z \bigcap (2\pi/s)Z] = 1$. If t/s is irrational, the two sets intersect only at the origin and $P[Y = 0] = 1$.

In particular, if the characteristic function of Y takes the value 1 on an interval, then $Y = 0$ almost surely and the characteristic function is 1 everywhere.

5 The Main Proof

The ingredients are now all in place for the proof.

As seen above, the convergence set C for the sequence of partial sums satisfies

$$P[C] = \lim_{k\to\infty} \lim_{m\to\infty} \lim_{n\to\infty} P[\max_{m<r\leq n} |S_r - S_m| \leq 1/k]$$

so almost sure convergence of the series will be established if for any $\varepsilon > 0$ the iterated limit

$$\lim_{m\to\infty}\lim_{n\to\infty} P[\max_{m<r\le n} |S_r - S_m| > 2\varepsilon] = 0.$$

Under the assumption that S_n converges in distribution, $\phi_{S_n}(t) = \prod_{j=1}^{n} \phi_{X_j}(t)$ converges to some characteristic function uniformly on compact subsets of the real line. Let S denote an unspecified random variable having this limiting characteristic function. Then there is a non-degenerate interval I containing the origin on which none of the characteristic functions $\phi_S, \phi_{S_1}, \phi_{S_2}, \ldots$ vanish. Simple computations then show that for $m \le n$, $\phi_{S_n - S_m}(t) = \phi_{S_n}(t)/\phi_{S_m}(t)$ for $t \in I$. So any limiting distribution of $S_n - S_m$, as both $m \le n \to \infty$, has a characteristic function which is 1 on I, and is hence 1 everywhere. Thus every limiting distribution of $S_n - S_m$ is degenerate at 0.

As a first step, consider computing the iterated limit of the right side of Ottaviani's inequality: $\lim_{m\to\infty}\lim_{n\to\infty} P[|S_n - S_m| > \varepsilon]$. Clearly there is no reason to believe that either limit exists, so compute instead the limsup in both cases. Denote by $L_m = \limsup_{n\to\infty} P[|S_n - S_m| > \varepsilon]$, and for each m find $n(m) > m$ so that $\left|L_m - P[|S_{n(m)} - S_m| > \varepsilon]\right| < 2^{-m}$. Now for each m let μ_m denote the probability measure induced by $S_{n(m)} - S_m$. By Helly's Selection Theorem, each subsequence of this family of measures has a further subsequence which converges to a measure μ which possibly has total measure less than one. On the interval I mentioned above, the charactersitic function of μ_m is $\phi_{S_{n(m)}}(t)/\phi_{S_m}(t)$ and thus the characteristic function of μ is 1 on I. By the foregoing discussion, this implies that μ is the probability measure supported on a single atom at 0. Thus $\lim_{m\to\infty} P[|S_{n(m)} - S_m| > \varepsilon] = 0$ and, since $|L_m - P[|S_{n(m)} - S_m| > \varepsilon]| < 2^{-m}$, $\lim_{m\to\infty} L_m = 0$ too. Therefore $\limsup_{m\to\infty}\limsup_{n\to\infty} P[|S_n - S_m| > \varepsilon] = 0$.

Now consider the same iterated limsup of the left hand side of Ottaviani's inequality. Each of the factors is monotone in n for fixed m and also monotone in m. So the iterated limsup is actually the iterated limit, and using what has been shown about the iterated limsup of the right hand side gives

$$\lim_{m\to\infty}\lim_{n\to\infty} P[\max_{m<r\le n} |S_r - S_m| > 2\varepsilon] \cdot \lim_{m\to\infty}\lim_{n\to\infty}\min_{m<r\le n} P[|S_n - S_r| \le \varepsilon] = 0.$$

Proceding as above will show that the second factor is 1 and that the first factor is therefore 0, which will complete the proof of almost sure convergence.

Denote by $L_m = \lim_{n\to\infty}\min_{m<r\le n} P[|S_n - S_r| \le \varepsilon]$. For each m select $n(m) \ge r(m) > m$ so that $|L_m - P[|S_{n(m)} - S_{r(m)}| \le \varepsilon]| < 2^{-m}$. Let μ_m be the probability measure induced by $S_{n(m)} - S_{r(m)}$ and argue as before to see that this sequence of probability measures converges to the probability measure supported at 0. Thus $\lim_{m\to\infty} P[|S_{n(m)} - S_{r(m)}| \leq \varepsilon] = 1$, which is also the value of the iterated limit, completing the proof.