Lecture Notes on Calculus

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§0. Preface

The objective of these notes is to present the basic aspects of calculus. As implied by the name, calculus contains many rules for calculation. These rules are indeed important, but computer software packages allow many of these calculations to be done by machine. The emphasis in these notes is on the concepts behind the computational formulas. These concepts give calculus its power and importance.

The specific objectives are the following.

(1) Develop a solid understanding of functions and the geometric and algebraic meaning of the graph of a function. Develop the ability to translate geometric properties of the graph of a function into equations, and vice-versa.

(2) Develop understanding, not just algorithms.

(3) Know the difference between an identity and an equation, and be able to use substitution rules stemming from identities correctly.

(4) Develop an intuitive understanding of limits.

(5) Develop an intuitive understanding of derivatives.

(6) Devlop an intuitive understanding of integrals.

(7) Develop the the ability to translate a verbal description of a problem into a mathematical description, and vice-versa.

Throughout these notes are various exercises and problems. The reader should attempt to work all of these. Solutions, sometimes in the form of hints, are provided for most of the problems.

These notes begin with a sweeping overview of calculus that captures the two most important ideas of the subject in a rather intuitive way. Then these two basic ideas are explored in a bit more depth together with the beginnings of a discussion of how these ideas are used in practice. Finally the two initial ideas are explored at nearly the ultimate level of detail and the discussion is broadened to include other related ideas.

§1. Mythology of Calculus

According to the dictionary, a myth is a traditional story serving to explain some phenomenon, often an aspect of nature. In this section a story is given which explains the essential ideas involved with calculus. The motivating observation is that straight lines are simple, while general curves are complicated. The central objective of calculus is to develop methods of using straight lines to study general curves.

Suppose $f(x)$ is a real valued function and imagine its graph. In fact, do more than imagine its graph–draw the graph of the function you imagine on a blank piece of paper. Select an arbitrary point a on the x axis and examine the corresponding point $(a, f(a))$ on the graph. View the graph at a normal distance. From this perspective you can see the behavior of the function for many values of x; this behavior can depend in an erratic way on x. Narrow your field of vision to a small region surrounding the point $(a, f(a))$ on the graph. Now only the details of the graph near $(a, f(a))$ can be viewed. Narrow your field of vision even more until only a small region the size of a pencil eraser near the point $(a, f(a))$ is visible. The part of the graph that you can still see appears to be a straight line. This is the first part of the calculus story:

For any function f and any point a on the x axis the graph of f near $(a, f(a))$ is approximately a straight line.

Exercise 1–1. Draw the graph of a function f for which there is a point a at which this statement is false. How did this happen?

Exercise 1–2. Choose a second point b on the x axis and repeat the visual experiment with the point $(b, f(b))$ on the graph. Are the slope and intercept of the line you see the same as before?

The line that is found by conducting the above experiment at the point $(a, f(a))$ will be called the **approximating line** to the graph of f at the point $(a, f(a))$. The collection of all of these approximating lines provides important information about the graph of f.

Select a dozen equally spaced points on the x axis and sketch short line segments to represent the approximating line at each of these points. Now take a second blank sheet of paper, place it atop the first sheet and sketch only the approximating lines on the new sheet. What function could have a graph with these lines as its approximating lines? It's easy to sketch in the graph of the function, at least near the points at which the approximating lines are drawn. This is the second part of the calculus story.

A function can be completely reconstructed from its approximating lines together with a single point known to be on the graph.

These two statements embody the central ideas of calculus. Taken together the two statements above form what might be called the calculus story:

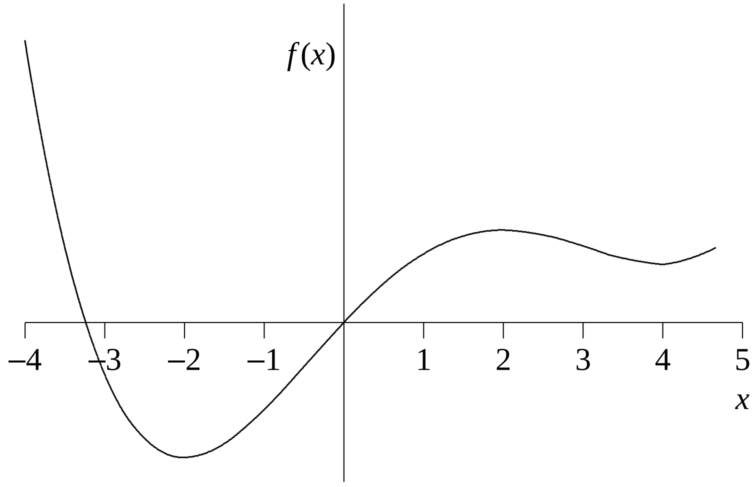
For any function f and any point a on the x axis the graph of f near $(a, f(a))$ is approximately a straight line. A function can be completely reconstructed from a given point on the graph and the slopes of all of these approximating lines.

The remainder of the study of calculus consists of making the ideas contained in the calculus story precise, and in developing techniques for putting these ideas to use.

As a prelude to a precise retelling of the calculus story, the next two sections contain foundational material on sets and functions.

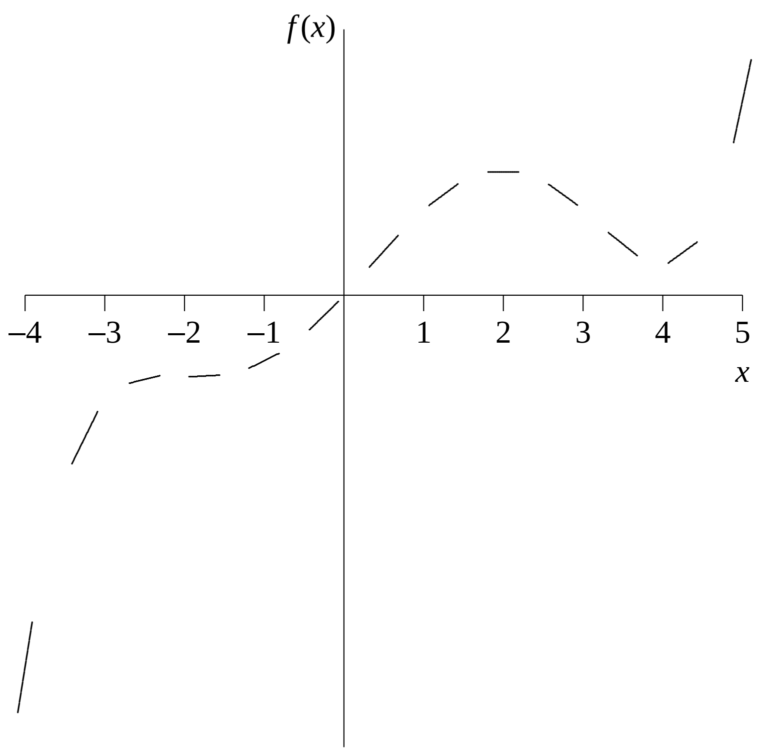
Problems

Problem 1–1. The graph of $f(x)$ is shown in the picture below. Sketch the approximating lines of f at the points $x = 1$, $x = 2$ and $x = 3$.



Problem 1–2. In the graph of f of the previous problem, at what values of x is the slope of the approximating line equal to 0? What is the distinguishing feature of these points?

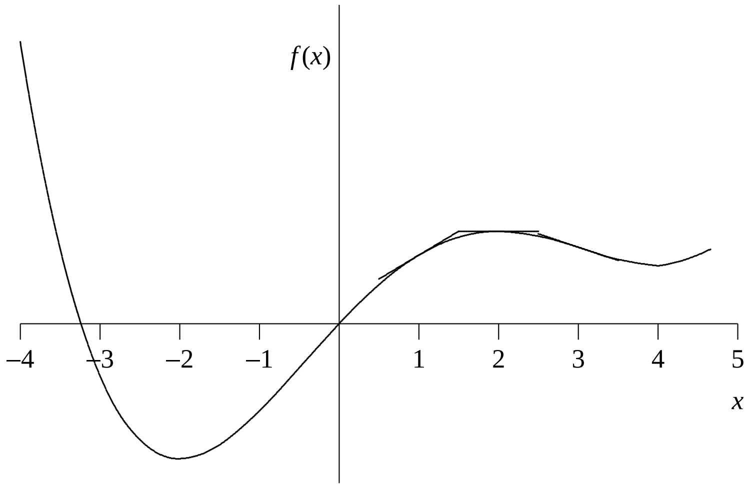
Problem 1–3. Some of the approximating lines of a function g are indicated below. Use this information to sketch the graph of g.



Problem 1–4. For a function h the slope of the approximating line is always negative. If $h(2) = 5$, could $h(12) = 17$? Why?

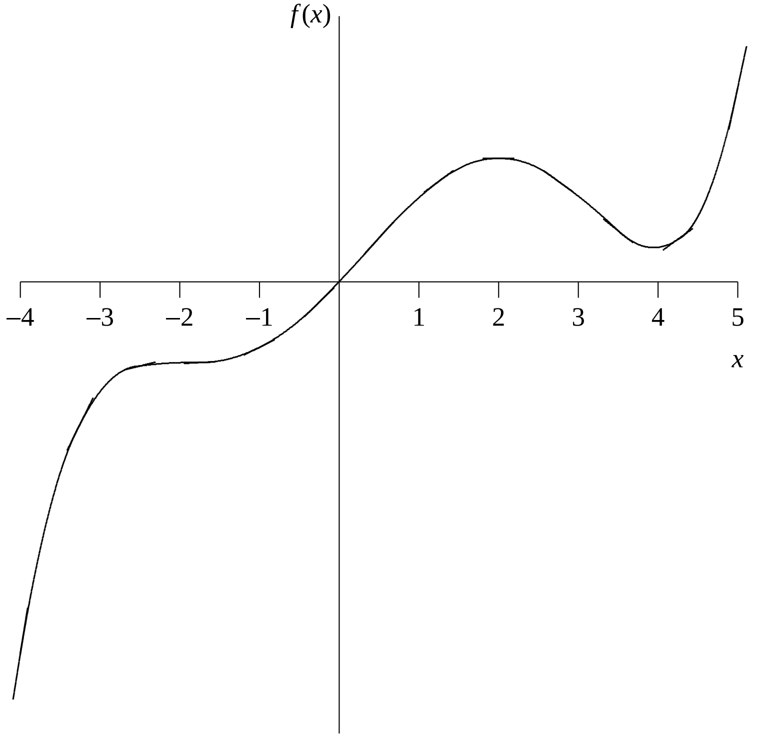
Solutions to Problems

Problem 1–1.



Problem 1–2. At $x = -2$, $x = 2$, and $x = 4$. These values of x mark the location of peaks or troughs in the graph of f.

Problem 1–3.



Problem 1–4. The function values must be getting smaller as x increases from 2 to 12, otherwise some approximating line would have positive slope. So $h(12) = 17$ is impossible.

Solutions to Exercises

Exercise 1–1. One possibility is a graph of a function having a jump-type break at a certain point. Near such a point the graph looks like two lines, not one. Are there other types of examples?

Exercise 1–2. The slope of the approximating line may change, as may the intercept. Neither must change, however. Can you give an example for which the slope and intercept of the approximating lines are the same at all points?

§2. Sets

Calculus has a strong geometric flavor. Geometric space is nothing more than a collection of points, and geometric objects in space consist of some sub-collection of points in space. The mathematical language of sets is used to provide an accurate description of geometric objects.

A **set** is simply a collection of objects. One might speak of the set of students in this classroom, the set of bicycles on campus, and so on. An individual object in a set is called an **element** of the set.

In mathematics, the sets of interest often consist of numbers. One of most often used sets is the set of real numbers. The collection of all numbers which can be written in decimal form (repeating or not) is the set of **real numbers**, and is denoted by **R**.

Giving sets a visual representation is often very useful. The visual representation of the set of real numbers **R** is as a straight, infinite, line. The individual numbers (elements) are located along this line.

Often, additional requirements are made which narrows the set of possible values to a piece, that is, a **subset**, of the original set. The subset is specified notationally by giving the condition required to be an element of the subset.

Example 2–1. The set of real numbers which are at least 3 is written notationally as $\{x \in \mathbf{R} : x \geq 3\}$. The notation is read as "the set of x in the real numbers such that x is greater than or equal to 3." In this notation the colon is read as "such that" or "with the property that." The notation $x \in \mathbf{R}$, which is read as "x is an element of **R**," means that the number x is an element of the set of real numbers. The inequality following the colon gives the additional property required to be a member of this particular subset. This same set could be written $\{x : x \in \mathbf{R} \text{ and } x \geq 3\}$.

Exercise 2–1. Give this set a visual interpretation by graphing it on a number line.

Exercise 2–2. Translate into words: $\{x : x \in \mathbf{R} \text{ and } x \leq 1/2\}$.

Exercise 2–3. Graph the set in the previous exercise.

Exercise 2–4. Is $5 \in \left\{x \in \mathbf{R} : x^2 > x\right\}$?

As the example suggests, subsets of the real line are often connected with algebraic problems involving a single variable. In many cases the relationship of interest will be between two or more variables. A higher dimensional set is used as the backdrop for visualizing such relationships.

For a situation involving two variables the subsets will be visualized in a two dimensional plane. Denote by $\mathbf{R}^2$ the set $\{(x,y) : x \in \mathbf{R} \text{ and } y \in \mathbf{R}\}$. This is the set of ordered pairs (x,y) in which each member of the pair is a real number. The two numbers are called the **coordinates** of the point. Visually, $\mathbf{R}^2$ is a two dimensional plane.

Example 2–2. The set $\left\{(x,y) \in \mathbf{R}^2 : x = 3 \text{ and } y = 5\right\}$ can be visualized easily. This set consists of a single point. As a notational convenience, this point is written as $(3,5)$.

Example 2–3. A basic way of visualizing a more complicated set, such as $A = \left\{(x,y) \in \mathbf{R}^2 : y = 2x\right\}$ is to first rewrite this set as $\{(x,2x) : x \in \mathbf{R}\}$ and then plot several individual points in the set, hoping to see a pattern. (This method is tedious for humans, but easy for computers.)

Exercise 2–5. What familiar geometric object is the set in the previous example?

Exercise 2–6. Is the set $B = \{(2x,4x) : x \in \mathbf{R}\}$ the same as the set A?

The previous exercise illustrates that the same set can have many different descriptions. Showing that two descriptions are describing the same set is accomplished by taking an arbitrary element meeting the first description and showing that this element also meets the second description, and vice-versa.

Example 2–4. Are the sets $\left\{(x,y) \in \mathbf{R}^2 : x - y = 5\right\}$ and $\{(t,t-5) : t \in \mathbf{R}\}$ the same? Suppose (x,y) is in the first set. Using the condition gives $y = x - 5$, so in fact $(x,y) = (x,x-5)$, and since x is a real number, this point meets the requirement to be an element of the second set. On the other hand, suppose $(t,t-5)$ is in the second set. Since $t - (t-5) = 5$, this point meets the requirement to be in the first set. Thus the two descriptions are describing the same set.

Exercise 2–7. Is the set $\{(s+5,s) : s \in \mathbf{R}\}$ the same as the set in the example?

Sometimes a set has no elements. The set with no elements is called the **empty set** and is denoted by $\emptyset$.

Example 2–5. The set $\left\{x \in \mathbf{R} : x^2 = -5\right\}$ has no elements, so $\left\{x \in \mathbf{R} : x^2 = -5\right\} = \emptyset$.

Spaces of dimension larger than 2 are often useful as well. Denote by $\mathbf{R}^d$ the set $\{(x_1,\ldots,x_d) : x_i \in \mathbf{R} \text{ for } 1 \le i \le d\}$. This is the set of ordered d-tuples of real numbers. Most of the work here will involve the spaces $\mathbf{R}^2$ and $\mathbf{R}^3$. The results and ideas can be easily carried over into spaces of higher dimension.

Problems

Problem 2–1. Find the set $\left\{x \in \mathbf{R} : \sqrt{x^2} = x\right\}$ and graph it.

Problem 2–2. Find the set $\{x \in \mathbf{R} : 2x + 3 = 5\}$ and graph it.

Problem 2–3. Find the set $\left\{x \in \mathbf{R} : (x + 2)^2 = x^2 + 4x + 4\right\}$ and graph it.

Problem 2–4. Write in set notation: the set of real numbers between 4 and 7, exclusive. What geometric object is this set?

Problem 2–5. Write in set notation: the set of points in the plane for which the second coordinate is 2 more than the first coordinate. What geometric object is this set?

Problem 2–6. True or False: $7 \in \left\{x \in \mathbf{R} : x^2 - 2x + 5 > 3\right\}$.

Problem 2–7. True or False: The point $(2, 3)$ is an element of the set

$$\left\{(x, y) \in \mathbf{R}^2 : xy - 5x = 7\right\}.$$

Problem 2–8. Are the sets

$$\left\{(x, y) \in \mathbf{R}^2 : 2x + 3y = 7 \text{ and } 4x + 6y = 14\right\}$$

and

$$\left\{(s, t) \in \mathbf{R}^2 : 6s + 9t = 21\right\}$$

the same? Geometrically, what are these sets?

Problem 2–9. Consider the region $S = \left\{(x, y) \in \mathbf{R}^2 : 0 \le x \le 2, 0 \le y \le x + 4\right\}$. Sketch the region S, and label each corner of the region with the coordinates of the corner point. What is the area of the region S?

Problem 2–10. Graph the set $\left\{(x, y) : 0 \le x \le 2, 0 \le y \le x^2\right\}$.

Solutions to Problems

Problem 2–1. $\left\{x \in \mathbf{R} : \sqrt{x^2} = x\right\} = \{x \in \mathbf{R} : x \geq 0\}$. The graph is the half line beginning at the 0 and extending to the right.

Problem 2–2. $\{x \in \mathbf{R} : 2x + 3 = 5\} = \{x \in \mathbf{R} : x = 1\}$ which graphs as a single point.

Problem 2–3. $\left\{x \in \mathbf{R} : (x+2)^2 = x^2 + 4x + 4\right\} = \mathbf{R}$. The equation $(x+2)^2 = x^2 + 4x + 4$ is an example of an **identity**, since equality holds for all values of x for which both sides are defined.

Problem 2–4. $\{x \in \mathbf{R} : 4 < x < 7\}$. This set is a line segment, without its endpoints.

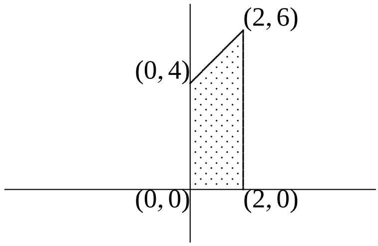
Problem 2–5. $\left\{(x, y) \in \mathbf{R}^2 : y = x + 2\right\}$ or $\{(x, x+2) : x \in \mathbf{R}\}$. This set is a line.

Problem 2–6. Here 7 is a real number and $7^2 - 2 \times 7 + 5 = 49 - 14 + 5 = 40 > 3$, so the answer is true.

Problem 2–7. Since $2 \times 3 - 5 \times 2 = -4 \neq 7$ the answer is false.

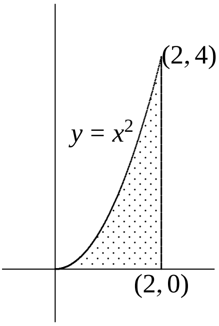
Problem 2–8. Yes. If (x, y) is in the first set, then $2x + 3y = 7$ and by multiplication the second requirement is also met. Multiplying by 3 shows that $6x + 9y = 21$, so the point (x, y) meets the requirement to be in the second set. If (s, t) is in the second set, then $6s + 9t = 21$ and division by 3 shows that $2s + 3t = 7$ while multiplication by $2/3$ shows that $4s + 6t = 14$ also holds. So the requirements for (s, t) to be in the first set are satisfied. Thus the two sets are the same. Geometrically, the sets are a line in $\mathbf{R}^2$.

Problem 2–9.



The area is the area of a rectangle plus the area of a triangle, and is $4 \times 2 + (1/2)2 \times 2 = 10$.

Problem 2–10. This set is the region in the plane bounded above by the parabola $y = x^2$, the x-axis, and the line $x = 2$.



Solutions to Exercises

Exercise 2–2. The set of real numbers that are less than or equal to $1/2$.

Exercise 2–3. The graph is an infinite ray which extends from the point $1/2$ to the left.

Exercise 2–4. The number 5 is a real number and $5^2 = 25 > 5$, so the answer is yes.

Exercise 2–5. A line through the origin.

Exercise 2–6. Yes, each point in B has a second coordinate which is twice its first coordinate.

Exercise 2–7. Yes. If $(s + 5, s)$ is in this new set, then since $s + 5 - s = 5$, this point meets the requirement to be in the first set of the example. On the other hand, if (x, y) is in the first set of the example, then $x = y + 5$ so that $(x, y) = (y + 5, y)$, and since y is a real number this point meets the requirement to be in the new set. Thus the new set and the first set of the example are the same. Can you give the argument to show that the new set and the second set of the example are the same?

§3. Functions

Calculus involves the study of functions. A function consists of three parts.

(1) A set called the **domain** of the function.

(2) A set called the **range** of the function.

(3) A rule which assigns to each element of the domain one and only one element of the range.

Often a function is specified just by giving the rule. In such cases the domain is then understood to be the largest set on which the rule makes sense, and the range is the set of output values that results by applying the rule to each of the elements in the domain.

Example 3–1. The rule $(x, y) \mapsto x + y$ defines a function with domain $\mathbf{R}^2$ and range space $\mathbf{R}^1$. (The symbol $\mapsto$ is read 'maps to.')

Usually functions are given a symbolic name which is attached to the rule.

Example 3–2. A function f can be defined by the formula $f(x, y) = x + y$.

Exercise 3–1. What is the domain and range of the function $g(x, y) = \sqrt{xy}$?

Example 3–3. What is the domain and range of the function $h(x, y) = (x + y, x - y)$? The formula defining h makes sense for any pair of input values, so the domain is $\mathbf{R}^2$. Any pair of output values are also possible, so the range is $\mathbf{R}^2$ as well.

Exercise 3–2. What pair of input values x and y would produce an output value of $(2, 3)$ for the function h?

Exercise 3–3. What is the domain and range of the function $k(x, y) = (x - y, x - y)$?

The **graph** of a function is the set of all possible pairs of elements for which the first element of the pair lies in the domain of the function and the second element of the pair is the corresponding output value. The graph must therefore lie in a space whose dimension is the sum of the dimensions of the domain and the range of the function.

Example 3–4. The graph of the function $f(x, y) = x + y$ is

$$\left\{(x, y, f(x, y)) : (x, y) \in \mathbf{R}^2\right\} = \left\{(x, y, x + y) : (x, y) \in \mathbf{R}^2\right\}.$$

Notice that the graph is a subset of 3 dimensional space.

Example 3–5. In what space does the graph of the function $g(x, y) = (2x, x - y)$ lie?

In most cases here, the functions of interest here will have some subset of **R** as their domain, and some subset of either **R** or $\mathbf{R}^2$ as their range. Notation such as $f : \mathbf{R} \to \mathbf{R}^2$ is used to denote a function whose domain is a subset of **R** and whose range is a subset of $\mathbf{R}^2$.

There are five basic ways to build new functions from existing ones. Four of the methods are based on direct application of the four basic arithmetic operations.

(1) The sum of two functions f and g is the function $f + g$ whose rule is $(f + g)(x) = f(x) + g(x)$. The domain of the sum consists of those elements which lie in both the domain of f and the domain of g. There is no convenient way of describing the range of $f + g$.

(2) The difference of two functions f and g is the function $f - g$ whose rule is $(f - g)(x) = f(x) - g(x)$.

(3) The product of two functions f and g is the function fg whose rule is $(fg)(x) = f(x)g(x)$.

(4) The quotient of two functions f and g is the function f/g whose rule is $(f/g)(x) = f(x)/g(x)$. The domain of f/g consists of those elements x that are in both the domain of f and the domain of g which also have the property that $g(x) \neq 0$.

The fifth way of combining two functions is to use the output of one function as input to another. If the range of the function g is contained in the domain of the function f, the **composition** of f with g, denoted $f \circ g$, is the function defined by the formula $(f \circ g)(v) = f(g(v))$. Notice that the function which is applied first is the one farthest to the right in the notation.

Example 3–6. Suppose f has domain and range **R** and is given by $f(x) = 2x - 3$ while g has domain **R** and is given by $g(x) = x^2$. Then $(f \circ g)(x) = 2x^2 - 3$. Notice that $(g \circ f)(x) = (2x - 3)^2$.

Occasionally, for a given function f there will be another function which ‘undoes what f does.’ This function is called the **inverse function** of f and is denoted by f^{-1}. The requirements for the inverse function are that

$f^{-1}(f(x)) = x$ for all x in the domain of f, and

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} (which is the range of f).

Graphically, the function f has an inverse function only if the graph of f passes the horizontal line test: each horizontal line intersects the graph of f in at most one point. Algebraically, the function f has an inverse function only if the equation $y = f(x)$ can be solved uniquely for x whenever y is in the range of f.

Example 3–7. If $f(x) = 2x - 3$ then $f^{-1}(x) = (x + 3)/2$. Simple substitution shows that the two requirements are met.

Problems

Problem 3–1. Write the graph of the function $f(x) = x^2$ in set notation.

Problem 3–2. True or False: If $f : \mathbf{R} \to \mathbf{R}^2$ the graph of f is a subset of $\mathbf{R}^2$.

Problem 3–3. True or False: For any real number x, $\sqrt{x^2} = x$.

Problem 3–4. True or False: If g has an inverse function and if $g(2) = 0$ then $g^{-1}(0) = 2$.

Problem 3–5. Consider the function $g(t) = (t + 5, t^2 - 7t)$. What is the domain of g? In what space does the range of g lie? In what space does the graph of g lie?

Problem 3–6. True or False: If $f : \mathbf{R}^2 \to \mathbf{R}$ it is possible that $f(2, 3) = (1, 7)$.

Problem 3–7. Suppose $g(t) = (t, t^2)$ and $f(x, y) = (x - y, x + y)$. What is $f \circ g$? What is $g \circ f$?

Problem 3–8. Suppose $f(t) = 1/(1 + t^2)$ for $t \geq 0$. Does f^{-1} exist? If so, find it.

Problem 3–9. Suppose $g(t) = (t, 3t)$. Does g^{-1} exist? If so, find it.

Solutions to Problems

Problem 3–1. The graph is $\{(s,t) \in \mathbf{R}^2 : t = s^2\}$. This could also be written as $\{(x,x^2) : x \in \mathbf{R}\}$.

Problem 3–2. False. The graph is a subset of $\mathbf{R}^3$.

Problem 3–3. False. The equality holds only for non-negative real numbers.

Problem 3–4. True.

Problem 3–5. The domain of g is $\mathbf{R}$. The range of g lies in $\mathbf{R}^2$, while the graph of g lies in $\mathbf{R}^3$.

Problem 3–6. False. The output value of f must be a number.

Problem 3–7. $(f \circ g)(t) = (t - t^2 t + t^2)$, while $g \circ f$ does not make sense.

Problem 3–8. Notice that the range of f is the set of real numbers y for which $0 < y \le 1$. Now if $y = 1/(1+t^2)$, solving for t gives $t = \sqrt{1/y - 1}$. Thus $f^{-1}(y) = \sqrt{1/y - 1}$.

Problem 3–9. The range of g is a line through the origin with slope 3. Since each value if t corresponds to exactly one point on the line, g^{-1} does exist. Now if (x,y) is a point in the range of g, the value of t that produces this point under the mapping g is x. Thus $g^{-1}(x,y) = x$. Notice that the formula $g^{-1}(x,y) = y/3$ is also valid. There are many others.

Solutions to Exercises

Exercise 3–1. The domain is $\{(x, y) : xy \geq 0\}$; the range is $\{x \in \mathbf{R} : x \geq 0\}$.

Exercise 3–2. The input values must satisfy $x + y = 2$ and $x - y = 3$, from which $x = 5/2$ and $y = -1/2$.

Exercise 3–3. The domain is $\mathbf{R}^2$, but the range is the line with equation $y = x$ in $\mathbf{R}^2$.

Exercise 3–3. The graph lies in 4 dimensional space.

§4. The Calculus Story Examined

In the next five sections the calculus story is examined in somewhat greater detail. In particular the problem of finding the equation of the approximating line is studied. This study leads to the restatement of the calculus story in symbolic form. In addition two physical interpretations are given to the slope of the approximating line. These interpretations are then used to develop some applications of the two fundamental ideas of calculus. The discussion culminates in the Fundamental Theorem of Calculus, which recasts the two aspects of the calculus story in a single equation.

§5. Approximating Lines and Reconstruction

In this section the equation of the approximating line of a function is examined in more detail. The calculus story is then retold in symbolic form. Once this is accomplished, techniques from algebra can be used for computations.

Given a function f and a point a on the x axis the equation of the line seen in the earlier visual experiment is to be computed. Evidently this line is determined by the points $(b, f(b))$ on the graph of f at points b which are near a. For convenience, write such points b as $b = a + s$ where s is a small number, that is, s is a number that is close to 0.

Example 5–1. Look at the simple example in which $f(x) = x^2$. What is the approximating line near $x = 3$? The objective is to study the graph of f near the point $(3, f(3)) = (3, 9)$. Now

$$f(3 + s) = (3 + s)^2 = 9 + 6s + s^2$$

for any value of s, small or not. The term $9 + 6s$ is recognizable as the equation of a line with slope 6 in the variable s. If s is close to zero, s^2 is much, much smaller than s. So for values of s close to zero, $f(3 + s)$ is very nearly equal to $9 + 6s$, that is, the values of $f(3 + s)$ nearly fall on the line with equaiton $9 + 6s$. So the slope of the approximating line is 6. Notice also that $f(3) = 9$, and the equation above has the form $f(3 + s) = f(3) + 6s + s^2$.

The slope of the approximating line to the graph of the function $f(x)$ at the point $x = a$ is called the **derivative** of the function f at the point $x = a$ and is denoted $f'(a)$.

Example 5–2. The previous example has shown that for the function $f(x) = x^2$, $f'(3) = 6$.

Example 5–3. For the function $f(x) = x^2$ the slope of the approximating line at an arbitrary point $x = a$ can be computed in a similar manner. For any value of s,

$$f(a + s) = (a + s)^2 = a^2 + 2as + s^2 = f(a) + 2as + s^2.$$

If s is near 0, s^2 is much smaller than s and this last expression is nearly equal to $f(a) + 2as$, which is the equation of a line with slope $2a$ (recall that s is the variable!). Thus $f'(a) = 2a$.

Exercise 5–1. What is the slope of $g(x) = x^3$ at $x = 2$? At $x = a$?

The previous exercise suggests that computing slopes in any particular case is a mechanical, though perhaps tedious, process. This is almost always true.

Fortunately there are some simple rules for carrying out this process. These rules will be developed later.

The preceding discussion allows the calculus story to be retold in symbolic form.

For any function f and any point a on the x axis the derivative $f'(a)$ is the number for which the equation

$$f(a+s) = f(a) + sf'(a) + \text{error small relative to } s$$

holds for all s near 0.

The reconstruction part of the calculus story is told symbolically by neglecting the terms small compared to s and using the approximation $f(a+s) \approx f(a) + sf'(a)$. This allows the graph of f to be approximately reconstructed from knowledge of the derivative by moving from left to right in small steps of size s from any given starting point. This method of reconstruction is called **Euler's method**, after the Swiss mathematician Leonhard Euler.

Example 5–4. To illustrate the reconstruction part of the story, suppose the function $e(x)$ has the property that $e'(x) = e(x)$ and $e(0) = 1$. To sketch the graph of $e(x)$ some points on the graph will be found. The fact that $e(0) = 1$ is given, so one point on the graph is known. Now select a small number s near 0, say $s = 0.1$. From the defining property of derivative with $a = 0$ and $s = 0.1$, $e(0.1) = e(0) + (0.1)e'(0)$ up to error that is small relative to s. Since $e'(0) = e(0) = 1$, this gives $e(0.1) = e(0) + e'(0)(0.1) = 1 + 1(0.1) = 1.1$, approximately. So a second point on the graph has been found. Applying the general formula again, this time with $a = 0.1$ and $s = 0.1$ gives $e(0.2) = e(0.1) + (0.1)e'(0.1) = 1.1 + (0.1)1.1 = 1.21$, approximately, since $e'(0.1) = e(0.1) = 1.1$ as found a moment ago. The results can be nicely summarized in a table.

x	$e(x)$	$e'(x)$
0	1	1
0.1	1.1	1.1
0.2	1.21	1.21
0.3		

This process can be continued to find as many points of the graph of $e(x)$ as required to make a nice sketch.

Exercise 5–2. Fill in the next row in the table of values of $e(x)$.

Exercise 5–3. What would happen in the example if $s = 0.01$ were used instead of $s = 0.1$?

There are some other notations that are used in connection with derivatives. The derivative of the function f at the point x and is denoted either by $f'(x)$ or $\frac{d}{dx}f(x)$. If the derivative is to be computed at a particular point, say $x = 5$, the notation for the derivative is either $f'(5)$ or $\left.\frac{d}{dx}f(x)\right|_{x=5}$.

Example 5–5. The computations of the earlier example show that $\frac{d}{dx}x^2 = 2x$.

Exercise 5–4. What is $\frac{d}{dx}x^3$?

As seen in the preceding exercise, the derivative of a function is itself a function, which again has a derivative. The derivative of the derivative of $f(x)$ is the **second derivative** of f and is denoted either by $f''(x)$ or $\frac{d^2}{dx^2}f(x)$. This process of computing additional derivatives can be continued indefinitely.

Example 5–6. If $f(x) = x^2$ then $f'(x) = 2x$, $f''(x) = 2$, and $f'''(x) = 0$.

Problems

Problem 5–1. What is $\frac{d}{dx}5$?

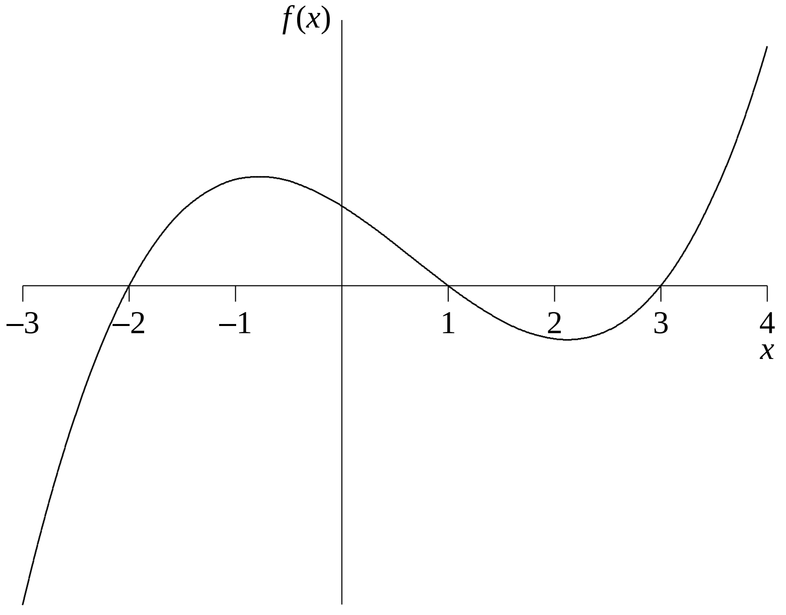
Problem 5–2. How are the derivatives of $2f(x)$ and $f(x)$ related? Is there anything special about '2'?

Problem 5–3. What is $\frac{d}{dx}(3x^2 + 7x + 5)$?

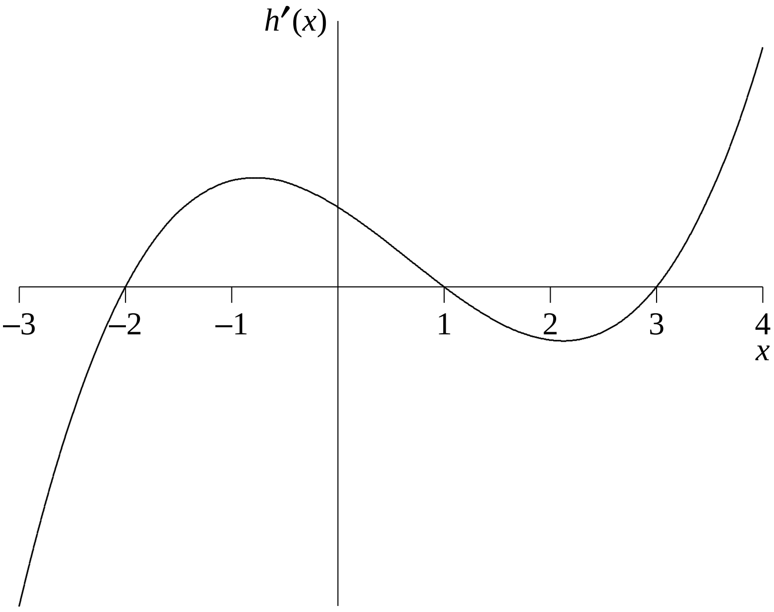
Problem 5–4. For a function $g(x)$ the slope of the approximating line at x is $1-g(x)$. If $g(0) = 0$ sketch the graph of $g(x)$ for $0 \le x \le 4$. You may wish to use spreadsheet software or a computer program.

Problem 5–5. Steel beams can be manufactured in any thickness t between 1 and 1,000 millimeters. The load bearing capacity of a beam increases continuously with the thickness of the beam. For $1 \le t \le 1000$, the load bearing capacity ranges from 0.1 to 15,000 kilograms. Denote by $L(t)$ the load bearing capacity, in kilograms, of a beam of thickness t, in millimeters. In the context of the information given, what is the domain of the function L? What is the range of L? Why does L have an inverse function? What is the physical interpretation, in this context, of the quantity $L^{-1}(500)$? If $L(3) = 0.2$ and $L'(3) = 0.5$, approximately what load would be supported by a beam 3.1 millimeters thick?

Problem 5–6. Below is the graph of a function $f(x)$. Sketch the graph of $f'(x)$.



Problem 5–7. Below is the graph of $h'(x)$. Sketch the graph of $h(x)$, assuming that $h(-3) = -1$.



Solutions to Problems

Problem 5–1. Here think of 5 as the function which takes the value 5 no matter what x is. Since the graph of this function is a horizontal line, the slope of the approximating line is always 0. So $\frac{d}{dx}5 = 0$.

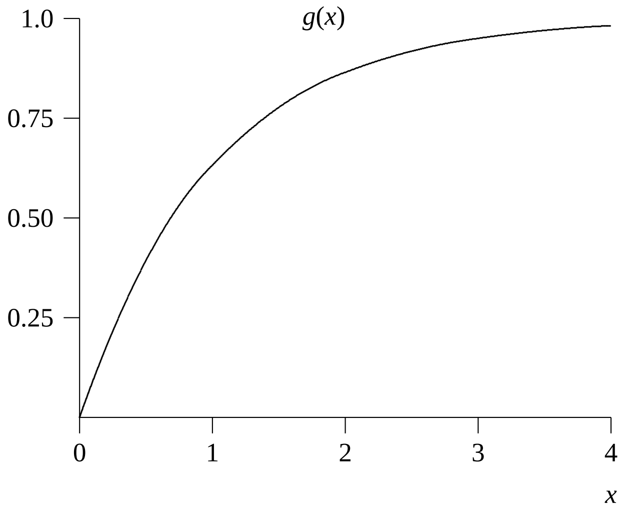
Problem 5–2. Computing gives

$$2f(a+s) - 2f(a) = 2(f(a+s) - f(a)) = s2f'(a) + \text{error small relative to } s$$

which is twice the slope of the approximating line for $f(x)$ at $x = a$. Thus $\frac{d}{dx}2f(x) = 2\frac{d}{dx}f(x)$. This argument works just as well for any number in place of the 2.

Problem 5–3. $6x + 7$. Is there a general rule for $\frac{d}{dx}(f(x) + g(x))$?

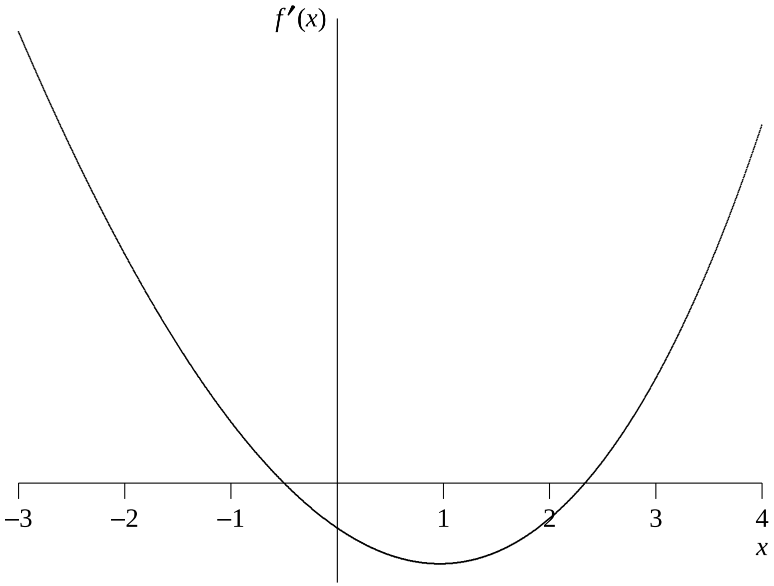
Problem 5–4. The graph looks something like this.



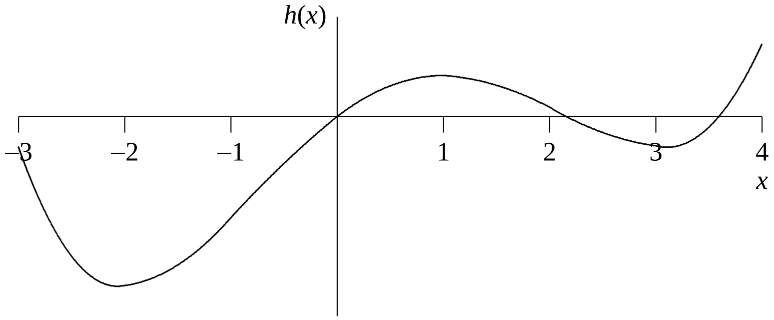
The computations were done as outlined in the last example of this section.

Problem 5–5. The domain of L is the set $\{t \in \mathbf{R} : 1 \le t \le 1000\}$. The range of L is the set $\{x \in \mathbf{R} : 0.1 \le x \le 15000\}$. Since the load bearing capacity increases with beam thickness, each load bearing capacity is associated with one, and only one, thickness. Thus if a load bearing capacity in the range of L is specified, the thickness corresponding to that capacity is uniquely determined. The quantity $L^{-1}(500)$ is the thickness of beam required to support a load of 500 kilograms. From the information given, $L(3.1) = L(3) + L'(3)(0.1) = 0.2 + (0.5)(0.1) = 0.25$ kilograms, approximately.

Problem 5–6.



Problem 5–7.



Solutions to Exercises

Exercise 5–1. At $x = 2$ the computation would be $g(2 + s) = (2 + s)^3 = 8 + 12s + 6s^2 + s^3 = g(2) + 12s + 6s^2 + s^3$. If s is near 0 this last expression is nearly equal to $g(2) + 12s$. So $g'(2) = 12$. Similarly $g'(a) = 3a^2$.

Exercise 5–2. $e(0.3) = e(0.2) + (0.1)e'(0.2) = 1.21 + (0.1)(1.21) = 1.331$. Do you agree that $e(0.4) = 1.4641$, approximately?

Exercise 5–3. The computed approximate values of the function would change slightly. The newly computed values would most likely be more accurate. A discussion of the error in using the reconstruction method will be given later.

Exercise 5–4. Using the earlier exercise and substituting x for a gives $\frac{d}{dx}x^3 = 3x^2$.

§6. Interpretations of the Derivative

There are two important interpretations of the derivative of a function f.

The first interpretation is geometrical and springs from the discussion above. The derivative of f at the point $x = a$ is the slope of the approximating line to the graph of f at the point $(a, f(a))$. This interpretation has many practical consequences.

Example 6–1. Suppose the derivative of a function g is always positive. The slope of the approximating line is therefore always positive. The values of the function g must get larger as x gets larger. This means that g is increasing: if $x < y$ then $g(x) < g(y)$.

The second interpretation of the derivative is as a rate of change. The reason for this interpretation can best be seen in an example.

Example 6–2. Suppose $D(t)$ denotes the position of a particle on the horizontal axis at time t. The derivative of $D(t)$ at time $t = a$ satisfies $D(a+s) = D(a) + sD'(t)$, neglecting the error term which is small relative to s. Solving this approximate equation gives

$$D'(t) = \frac{D(a+s) - D(a)}{a+s-a}$$

approximately, for any s near, but not equal to, 0. The numerator of this ratio is approximately the change in the particle's position between times a and $a+s$. The denominator is the elapsed time. The ratio is thus the change in position per unit time, that is, the velocity of the particle at time $t = a$.

Exercise 6–1. Why is the numerator only approximately the change in the position of the particle between times a and $a+s$?

Generally, the derivative $f'(x)$ can be interpreted as the rate at which the values of the function f are changing per unit change in x when x is the value of the independent variable.

Either of these two interpretations can be used in the context of a given problem. Often one of the interpretations provides a way of translating a verbal problem into a mathematical problem.

Problems

Problem 6–1. If $D(t) = t^2 - 4t$ for $t \geq 0$ is the position of a particle on the x axis at time t, for what values of t is the velocity positive? Negative?

Problem 6–2. Sketch the graph of the velocity of the particle of the previous problem.

Problem 6–3. What is the acceleration of the particle? (The acceleration is the rate at which the velocity is changing.)

Problem 6–4. Suppose $V(t)$ is the volume of water in a reservoir at time t. Give a verbal interpretation of $V'(t)$. How would the fact that the reservoir is filling with water be expressed mathematically?

Problem 6–5. The demand $D(p)$ for a good depends on the price p of the good. Give a verbal interpretation of $D'(p)$. What is the expected sign of $D'(p)$?

Problem 6–6. Wealth is measured in dollars, but the usefulness (or utility) of w dollars may be different for different people. Suppose $U(w)$ is the utility of total wealth w to a particular person. How is the fact that this person feels that more wealth is more useful expressed mathematically? As this persons wealth increases, the usefulness of an extra dollar declines. How is this fact expressed mathematically?

Problem 6–7. The fuel consumption, in gallons per hour, of a car traveling at speed v in miles per hour is $C(v)$. What is the meaning of $C'(v)$? What is the meaning of $v/C(v)$?

Solutions to Problems

Problem 6–1. The graph of $D(t)$ is a parabola which opens up and has vertex at $t = 2$. The velocity is positive if $t > 2$ and negative if $t < 2$.

Problem 6–2. On the one hand $D'(t) = 2t - 4$ which graphs as a straight line. However one can also proceed qualitatively as follows. First graph $D(t)$ and several of its approximating lines. Then eyeball the slopes of these approximating lines and sketch the graph of these slopes versus t. Unless your eyeball is really good, you won't get exactly a straight line, but you will see qualitatively what the velocity is.

Problem 6–3. The accelaration is 2.

Problem 6–4. $V'(t)$ is the rate at which the volume of the water in the reservoir is changing in time. If the reservoir is filling, then $V'(t) > 0$.

Problem 6–5. $D'(p)$ is the rate at which demand changes with price. Ordinarily, as price increases, demand decreases. One would expect $D'(p) < 0$.

Problem 6–6. In the first case, $U'(w) > 0$. In the second $U''(w) < 0$. Here $U''(w)$ is the second derivative of U, that is, $U''(w)$ is the derivative of $U'(w)$.

Problem 6–7. Here $C'(v)$ is the change in fuel consumption per unit increase in speed when the speed is v. The quantity $v/C(v)$ is the gas mileage, in miles per gallon, when the speed is v.

Solutions to Exercises

Exercise 6–1. If the times a and $a+s$ are far apart, the particle may have wandered far from the origin between these times. This fact would not be accounted for in $D(a+s)$ and $D(a)$, since these are the position at these two times only and do not reflect the motion of the particle in between these times.

§7. Applications of the Derivative

Many of the important applications of the derivative are found in the process of constructing mathematical models of physical phenomena. Indeed, calculus was invented largely to provide a mathematical basis for physics. This modeling process is examined here in a few relatively simple cases.

The first applications will be to the physics of motion. The essential physical ideas are the following. Objects possess a property called mass. Forces, which have both magnitude and direction, cause objects to move. Suppose a one dimensional coordinate system is fixed and denote by $D(t)$ the position of an object in this coordinate system at time t. The velocity $v(t)$ of the object at time t is defined to be the derivative of $D(t)$. The acceleration $a(t)$ of the object at time t is defined to be the derivative of $v(t)$. The momentum of an object at time t is defined to be the product of its mass and velocity at time t. The fundamental physical law is Newton's equation of motion:

$$\text{Net Force at time } t = \text{Rate of Change of Momentum at time } t.$$

The right side of this equation clearly involves a derivative.

Example 7–1. A ball of mass 5 kilograms is dropped from a tall building. Neglecting air resistance, the only force acting on the ball is gravity. Take the origin of the coordinate system to be at the top of the building, and the positive direction to be down. If g is the gravitational constant then Newton's equation of motion is

$$5g = \frac{d}{dt}5v(t).$$

Since $\frac{d}{dt}5v(t) = 5\frac{d}{dt}v(t) = 5a(t)$, this equation can be written as $a(t) = g$. The ball experiences a constant acceleration.

The equation of motion in the previous example is called a **differential equation** because the equation involves the derivative of an unknown function (in this case, $v(t)$). A large number of equations arising in physics and engineering, and many other disciplines, are differential equations. Often these equations can be solved to give an explicit formula for the unknown function. In other cases, the solution can only be sketched directly from the equation. In any case, the second part of the calculus story plays a prominent role.

Exercise 7–1. Sketch the acceleration, velocity, and distance functions for the ball. Since the ball is dropped, $v(0) = 0$. (The gravitational constant g is approximately 9.8 meters per second per second.)

Example 7–2. A more realistic model is obtained by including a term reflecting air resistance. One simple model is that the magnitude of the force of air resistance is proportional to the velocity. The constant of proportionality reflects the aerodynamic properties of the object. For simplicity, suppose this constant is 2. The direction of the air resistance force is always in the opposite direction of the velocity. With the same conventions as before the equation of motion becomes

$$5g - 2v(t) = \frac{d}{dt}5v(t).$$

The motion of the ball is quite different in this case.

Exercise 7–2. What is the acceleration in this case?

Exercise 7–3. Sketch the graph of the velocity as a function of time. Assume that $v(0) = 0$.

The preceding examples have illustrated how a differential equation can be derived from a general physical principle. The next example illustrates how a differential equation can arise from the analysis of a given situation. This example uses one of the most important techniques in the application of mathematics. The technique is sometimes referred to as the method of slicing.

Example 7–3. A tank contains 500 liters of pure water at time $t = 0$. Starting at time $t = 0$, 30 liters per minute of a 20% alcohol solution enters the top of the tank. Also starting at time $t = 0$, 30 liters per minute of the mixed solution is drained from the bottom of the tank. Suppose $C(t)$ is the fraction of the solution in the tank which is alcohol at time t. Since the tank always contains 500 liters of liquid, $500C(t)$ is the amount of alcohol in the tank at time t. For small s,

$$500C(t+s) = 500C(t) + (30)(0.2)s - (30)C(t)s + \text{error small compared to } s$$

by considering the amount of alcohol entering and leaving the tank in the short time interval from t to $t+s$. Thus $C'(t) = 6/500 - (30/500)C(t)$ is the differential equation governing the fraction of the solution which is alcohol at time t.

Exercise 7–4. For large values of t, what is the approximate value of $C(t)$?

Problems

Problem 7–1. In the dropped ball example suppose the magnitude of the air resistance force is $10\sqrt{v(t)}$. Write the equation of motion. Assume $v(0) = 0$. Sketch the velocity as a function of time. Could you use spreadsheet software to make a sketch?

Problem 7–2. In the previous problem how do things change if $v(0) = 200$?

Problem 7–3. Both the area $A(r)$ and the circumference $C(r)$ of a circle depend on the radius r. What is the relationship between $A(r)$ and $C(r)$? Hint: There is a differential equation lurking here. Draw two concentric circles, one with radius r and the other with slightly larger radius $r+s$ and look at the difference of their areas.

Problem 7–4. The rate of compound interest paid by an investment changes with time. At time t the interest rate is $\delta(t)$. The investment is worth 100 at time $t = 0$ and only increases in value due to deposits of interest. How are the value $V(t)$ of the account at time t and the interest rate related?

Problem 7–5. At time $t = 0$ a large tank is filled with 1000 liters of pure water. At time $t = 0$ two pipes begin to add liquid at the top of the tank. One of the pipes continuously adds 20 liters per minute of a solution which is 5% alcohol by volume; the other pipe continuously adds 30 liters per minute of a solution which is 40% alcohol by volume. The liquid in the container is continuously and vigorously stirred. At time $t = 0$ a pipe at the bottom of the container starts continuously withdrawing 50 liters per minute of the mixture from the container. Denote by $C(t)$ the fraction of the solution in the tank which is alcohol at time t. What is $C(0)$? Write the equation expressing the relationship between $C'(t)$ and $C(t)$ implied by the given facts. What is the approximate fraction of the solution in the tank that is alcohol after the mixing process has been in operation for a long time?

Solutions to Problems

Problem 7–1. The equation of motion is

$$5g - 10\sqrt{v(t)} = \frac{d}{dt}5v(t).$$

Your sketch should show that the velocity never exceeds $(5g/10)^2 \approx 25$.

Problem 7–2. The velocity decreases from 200 initially to 25 as time goes on.

Problem 7–3. From the picture of the two concentric circles, $A(r+s) - A(r) = C(r)s$ is approximately true. Thus $A'(r) = C(r)$ is the relationship.

Problem 7–4. At time $t + s$ which is slightly greater than t it must be approximately true that $V(t+s) - V(t) = V(t)\delta(t)s$. Hence $V'(t) = \delta(t)V(t)$.

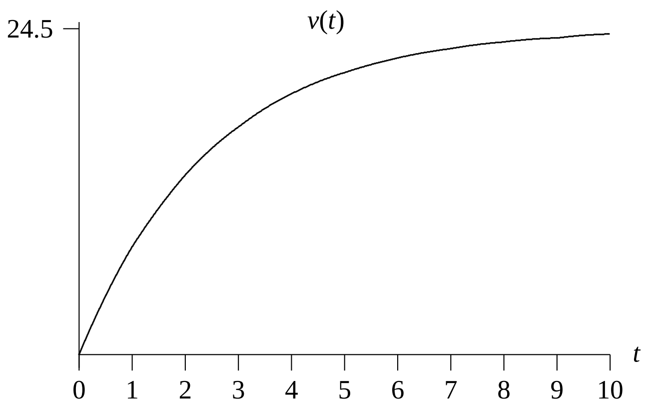
Problem 7–5. $C(0) = 0$, since the tank is initially filled with pure water. Here $1000C(t)$ is the amount of alcohol in the tank at time t. Thus $1000C'(t) = (0.05)(20) + (0.40)(30) - 50C(t)$ or $C'(t) = 0.013 - 0.05C(t)$. After a long time, the process will reach equilibrium at a fraction C for which $C'(t)$ is nearly zero. This fraction is $0 = 0.013 - 0.05C$ or $C = 13/50 = 0.26$.

Solutions to Exercises

Exercise 7–1. Since $a(t) = g = 9.8$, the acceleration function is a constant function which has a horizontal line as its graph. Since $v'(t) = a(t)$, the approximating lines for the velocity function all have the same slope and the same intercept. The graph of $v(t)$ is a line through the origin with slope 9.8. The distance funtion is a parabola with vertex at the origin.

Exercise 7–2. The acceleration is $a(t) = g - (2/5)v(t)$.

Exercise 7–3. Using $g = 9.8$ a table of values can be constructed as before. The graph looks like this.



The horizontal asymptote is at level $5g/2 \approx 24.5$. Can you explain why?

Exercise 7–4. For large values of t, the fraction of alcohol in the tank reaches equilibrium on physical grounds. Thus $C'(t) \approx 0$ for large t. Using the differential equation and this fact shows that $C(t) \approx 6/30 = 0.20$ for large t, in accord with physical intuition.

§8. Examining the Reconstruction Process

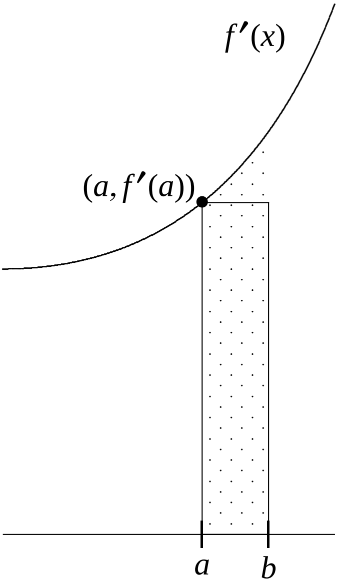
The reconstruction process is viewed in a geometric way which connects the reconstruction process to the computation of area. The definite integral is introduced to represent this area symbolically.

The second part of the calculus story is summarized symbolically by the approximate equation $f(a+s) = f(a) + sf'(a)$ which holds provided s is near 0. This equation provides a useful way of numerically completing the reconstruction process. The objective now is to understand this reconstruction process in a way that is conceptually simpler.

To begin, consider the graph of the function $f'(x)$ and select two points $b > a$ which are near each other on the x axis. The situation is illustrated in the picture below. Since a and b are near each other, the area of the region between the x axis and the curve $f'(x)$ for $a \le x \le b$ is approximately the same as the area of the rectangle with vertices at $(a,0)$, $(b,0)$, $(b, f'(a))$ and $(a, f'(a))$. The area of this rectangle is $f'(a)(b-a)$. Since $b-a$ is small, the difference $b-a$ can be used in place of s in the defining formula for the derivative. This substitution gives $f'(a)(b-a) = f(b) - f(a)$, approximately. Thus for b near a

$$\begin{aligned}\text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } a \le x \le b &= f'(a)(b-a)\\ &= f(b) - f(a).\end{aligned}$$

What happens if a and b are not near?



Suppose a and b are just a little too far apart to be called near, but there is a point c, $a < c < b$ which is near to both a and b. Reasoning as before then gives

$$\begin{aligned}
&\text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } a \le x \le b \\
&\quad = \text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } a \le x \le c \\
&\qquad + \text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } c \le x \le b \\
&\quad = f'(a)(c-a) + f'(c)(b-c) \\
&\quad = (f(c)-f(a)) + (f(b)-f(c)) \\
&\quad = f(b)-f(a).
\end{aligned}$$

This is the same result as before! If a and b are even further apart, more intermediate points can be introduced and the same argument can be repeated. The general conclusion is that

$$\text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } a \le x \le b = f(b)-f(a)$$

whether a and b are close or not!

Exercise 8–1. Suppose two intermediate points are needed. Verify that the same argument can be extended to this case and gives the same result.

The problem of reconstructing the function $f(x)$ from knowledge of its derivative $f'(x)$ is the same as the problem of finding the area between the graph of the derivative and the x-axis between any two points on the axis. Notice that the computational method shows that the area found is a *signed* area: intervals over which the graph of $f'(x)$ lies below the axis make a negative contribution to the computation of this area.

Some terminology and notation have been developed to make discussing the computation of areas symbolically easier. If $g(x)$ is any function the area between the graph of $g(x)$ and the x-axis for $a \le x \le b$ is called the **definite integral** of $g(x)$ for x between a and b and is denoted $\int_a^b g(x)\,dx$. In this notation, a is the **lower limit of integration** and b is the **upper limit of integration**. Also the function g appearing inside the integral is the **integrand**. The dx in the notation is simply a reminder that x is the variable of integration.

The computational formula developed above, and this discussion, is summarized in the **Fundamental Theorem of Calculus**: For any function f and any points a and b

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This theorem plays a central role in the discussion that follows.

A final note on the notation is this. If $a > b$ then $\int_a^b g(x)\,dx = -\int_b^a g(x)\,dx$. This is in keeping with both the Fundamental Theorem and the fact that the integral represents a signed area.

Problems

Problem 8–1. Compute $\int_0^3 2x\,dx$ by using the Fundamental Theorem of Calculus and also by finding the area of a triangle.

Problem 8–2. Suppose $h'(x) = f'(x)$ for all x. Is $\int_a^b f'(x)\,dx = h(b) - h(a)$?

Problem 8–3. $\int_{-3}^3 2x\,dx$ by using the Fundamental Theorem of Calculus and also by finding the area of two triangles. What is going on?

Problem 8–4. What is the geometric interpretation of $\int_{-3}^3 |2x|\,dx$ and how does this compare with $\int_{-3}^3 2x\,dx$?

Problem 8–5. True or False: If the function g has domain $\mathbf{R}$ and if g has an inverse function, then the graph of g^{-1} is the set $\{(g(x), x) : x \in \mathbf{R}\}$.

Problem 8–6. True or False: $\int_{-3}^7 \frac{t^2+1}{t^8+1}\,dt > 0$.

Solutions to Problems

Problem 8–1. Since $\frac{d}{dx}x^2 = 2x$ the Fundamental Theorem gives $\int_0^3 2x\,dx = 3^2 - 0^2 = 9$.

Problem 8–2. Since $h'(x) = f'(x)$ for all x, the areas determined by the two graphs are the same.

Problem 8–3. Computations involving the Fundamental Theorem treat 'areas' below the axis as though they were negative.

Problem 8–4. The presence of the absolute values restores the interpretation as geometrical area of the region between the graph of the function and the axis.

Problem 8–5. True, since $g^{-1}(g(x)) = x$ for all x in the domain of g.

Problem 8–6. True, since the graph of the integrand lies entirely above the horizontal axis. Thus the area represented by the integral must be positive.

Solutions to Exercises

Exercise 8–1. Suppose $a < c < d < b$ and the adjacent points are deemed to be near.

$$\begin{aligned}
&\text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } a \le x \le b \\
&\quad = \text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } a \le x \le c \\
&\qquad + \text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } c \le x \le d \\
&\qquad + \text{Area between the graph of } f'(x) \text{ and the } x\text{-axis for } d \le x \le b \\
&\quad = f'(a)(c-a) + f'(c)(d-c) + f'(d)(b-d) \\
&\quad = (f(c) - f(a)) + (f(d) - f(c)) + (f(b) - f(d)) \\
&\quad = f(b) - f(a).
\end{aligned}$$

§9. Interpretations of the Definite Integral

There are two important interpretations of the definite integral.

The first interpretation is the immediate by product of the previous discussion. The integral $\int_a^b g(x)\,dx$ of a function $g(x)$ from $x = a$ to $x = b$ is the signed area of the region bounded by the graph of the function and the x-axis for $a \le x \le b$. This signed area differs from the geometrical area of the region in that in the computation of the definite integral any portions of the region that lie below the axis are taken to have negative area.

Exercise 9–1. How can the geometric area of the region bounded by the graph of $g(x)$ for $a \le x \le b$ be written as an integral?

The second interpretation of the definite integral $\int_a^b g(x)\,dx$ is as the accumulated rate of change, that is, the total amount change for $a \le x \le b$ when the rate of change is $g(x)$. This interpretation can more easily be seen in the context of an example.

Example 9–1. In the falling ball example consider earlier let $D(t)$ be the total distance the ball has fallen by time t. Then $D'(t)$ is the downward velocity of the ball at time t, that is, the rate of change of the distance fallen. By the Fundamental Theorem, $\int_1^2 D'(t)\,dt = D(2) - D(1)$. Now $D(2) - D(1)$ is the total distance the ball falls between time $t = 1$ and $t = 2$. So $\int_1^2 D'(t)\,dt$ is the accumulated rate of change of distance fallen.

Example 9–2. The area between the graph of $f'(x)$ and the x-axis for $a \le x \le b$ lies between the area of two rectangles. The larger rectangle has the interval $a \le x \le b$ as its base and has height given by the maximum value of $f'(x)$ on the interval. The smaller rectangle has the interval $a \le x \le b$ as its base and height given by the miniumum value of $f'(x)$ on the interval. Geometric reasoning shows that, in fact, there is a rectangle having the interval $a \le x \le b$ as its base and height equal to *some* value of $f'(x)$ which has the same area as $\int_a^b f'(x)\,dx$. If c is the number $a \le c \le b$ for which $f'(c)$ is the height of this rectangle, then $f'(c)(b - a) = \int_a^b f'(x)\,dx$. This equality is one version of the **Mean Value Theorem**. A second verion is obtained by applying the Fundamental Theorem to the right side of the equation to obtain $f'(c)(b - a) = f(b) - f(a)$. The Mean Value Theorem is of great importance in the theoretical development of calculus.

Problems

Problem 9–1. Suppose $D(t)$ is the position of a particle on the horizontal axis at time t. What is the difference in interpretation between $\int_0^5 D'(t)\,dt$ and $\int_0^5 |D'(t)|\,dt$?

Problem 9–2. Suppose the region bounded by the graph of x^3 and the x-axis for $-3 \le x \le 3$, with x measured in feet, is to be painted. How much paint is required if one gallon of paint covers 200 square feet?

Problem 9–3. Suppose $C(t)$ is the rate of cash inflow to a corporate bank account at time t. Negative values of $C(t)$ represent cash outflow. If the account had a balance of 100,000 at time $t = 0$, write an expression for the amount of cash in the account at time $t = 5$.

Problem 9–4. Erin has constructed a simple model for water flow into and out of a lake. She defined the following functions.

(1) $V(t)$ is the volume, in liters, of water in the lake at time t.

(2) $R(t)$ is the rate, in liters per hour, at which water is entering the lake from a river at time t.

(3) $C(t)$ is the rate, in liters per hour, at which water is exiting the lake through a creek at time t.

(4) $E(t)$ is the rate, in liters per hour, at which water is evaporating from the lake at time t.

She believes that there are no other significant inflows or outflows of water from the lake. Time $t = 0$ is taken to be noon January 1, 2005 and time is measured in hours. Write an expression for $V'(t)$ in terms of the other functions. Write an expression for the total amount of water lost through evaporation between time 0 and 24. In this context, what is the physical interpretation of $\int_5^7 R(t)\,dt$? If $R(5) = 123$, compute an approximate value for $\int_5^7 R(t)\,dt$. Explain the physical conditions, in this context, under which your approximate answer is reasonable.

Solutions to Problems

Problem 9–1. $\int_0^5 D'(t)\,dt$ is the change is position between time 0 and 5 while $\int_0^5 |D'(t)|\,dt$ is the total distance the particle has travelled between time 0 and 5. These two quantities may be different!

Problem 9–2. The amount of paint required is $\int_{-3}^3 |x^3|\,dx/200$ gallons. Notice that $\int_{-3}^3 |x^3|\,dx = 2\int_0^3 x^3\,dx = 162/4$.

Problem 9–3. The amount at time $t = 5$ is $100,000 + \int_0^5 C(t)\,dt$.

Problem 9–4. Since there are no other significant inflows or outflows, $V'(t) = R(t) - C(t) - E(t)$. The evaporative loss is $\int_0^{24} E(t)\,dt$ liters. The integral is the volume of water, in liters, that entered the lake from the river between times 5 and 7. Since the integral represents the area between the graph of $R(t)$ and the axis, this area should be approximately the area of a rectangle with height $R(5)$ and width $7-5 = 2$. So the integral is approximately $123 \times 2 = 246$. If the river inflow is nearly constant this approximation should be reasonable.

Solutions to Exercises

Exercise 9–1. Write the geometric area as $\int_a^b |g(x)|\,dx$.

§10. Progress Report and a Look Ahead

The previous five sections complete a second look at calculus. The original calculus story was turned into symbolic form, and this symbolic form gave rise to formulas that were useful for computing the slope of the approximating line if the function itself was given, as well as for reconstructing the graph of the function if the slopes and intercepts of all the approximating lines were given. The derivative was seen to have two interpretations. These interpretations were useful in applying the ideas of calculus in some practical situations. The reconstruction problem led to the idea of the definite integral, which was also seen to have two interpretations. The Fundamental Theorem of Calculus connected in an explicit way the concept of derivative and integral.

In the next few sections, computational methods are developed for computing both derivatives and definite integrals.

§11. A Refinement of the Fundamental Equation

In order to develop computational methods that are correct, the error term in the fundamental equation

$$f(a+s) = f(a) + sf'(a) + \text{error small relative to } s$$

must be understood in a more precise way. What exactly does it mean to say that a quantity that depends on s is small relative to s?

Example 11–1. The quantity s^2 is small relative to s because the ratio $s^2/s = s$ is small whenever s is small (but s is not zero!).

Example 11–2. The quantity $s^{1/3}$ is not small relative to s, since $s^{1/3}/s = 1/s^{2/3}$ which is a large positive number when s is small. Note again that the ratio is not defined when $s = 0$.

The examples raise two key issues. First, stating that a quantity is small relative to s means that the quantity divided by s must be small whenever s is close enough to 0. Second, the value of the quantity when $s = 0$ is irrelevant to this discussion, since the ratio is not defined when $s = 0$.

To avoid writing the phrase 'small relative to s' repeatedly in the discussions that follow, any quantity that is small relative to s will be denoted by the symbol $o(s)$, which is read 'little-o of s.' This notation was popularized by the mathematician Edmund Landau in his 1909 book *Handbuch der Lehre von der Verteilung der Primzahlen*.

Example 11–3. In the first example above, the fact that s^2 is small relative to s would be written $s^2 = o(s)$.

Keep in mind that the symbol $o(s)$ may represent different quantities each time the symbol is written. The important fact is that $o(s)$ always represents a quantity that is small relative to s.

Example 11–4. Since $s^2 = o(s)$ and $s^3 = o(s)$, $s^2 + s^3 = o(s)$ too. More generally, $o(s) + o(s) = o(s)$. This last equation translates into the phrase that the sum of two quantities which are each small relative to s is itself small relative to s.

Using this notation, the fundamental equation becomes

$$f(a+s) = f(a) + sf'(a) + o(s).$$

In this form the equation can be easily manipulated algebraically in order to derive computational formulas.

How small does s have to be in order for $o(s)/s$ to be near zero? There is no real answer to this question. The important point is that $o(s)/s$ can be *forced* to be as close to zero as desired simply by requiring s to be near (but not equal to) zero.

Example 11–5. The assertion that $s^3 = o(s)$ is correct because the ratio $s^3/s = s^2$ can be forced to be close to zero by requiring that s is near zero. If someone wants to force $|s^3/s| < 0.0001$, all that is necessary is to require $0 < |s| < 0.01$.

Exercise 11–1. What requirement on s will force $|s^3/s| < 0.000001$?

Exercise 11–2. What requirement should be made on s to force $|s^5/s| < 0.0001$?

This notion that function values can be forced to be near a particular number by imposing requirements on the independent variable is embodied in the concept of limit. The **limit** of $h(x)$ as x approaches a is L, denoted $\lim_{x \to a} h(x) = L$, if the values $h(x)$ can be forced to be near L by requiring x to be near, but not equal to, a. The notion of limit will be explored in more detail later. The key observation for now is that $\lim_{s \to 0} o(s)/s = 0$.

Problems

Problem 11–1. Is it true that $o(s) - o(s) = 0$?

Problem 11–2. True or False: $100o(s) = o(s)$.

Problem 11–3. True or False: $so(s) = o(s)$.

Problem 11–4. True or False: $o(s)/s = o(s)$.

Problem 11–5. The notation $O(s)$, read 'big-o of s,' is used to denote a quantity that is about the same size as s, for s near 0. Stated more precisely, a quantity is $O(s)$ provided that the absolute value of the quantity is smaller than a fixed number times s, for s near 0. Is it true that $O(s) + O(s) = O(s)$? Is $sO(s) = o(s)$?

Solutions to Problems

Problem 11–1. Not necessarily. For example, each of s^2 and s^3 are $o(s)$, but the difference $s^2 - s^3$ is not zero. The difference *might* be zero, but generally all that can be asserted is that $o(s) - o(s) = o(s)$.

Problem 11–2. True. Multiplying a quantity that is small relative to s by a number produces a quantity that is still small relative to s.

Problem 11–3. True.

Problem 11–4. False. While $s^2 = o(s)$, $s^2/s = s$ is not small relative to s.

Problem 11–5. Both statements are true.

Solutions to Exercises

Exercise 11–1. In this case requiring $0 < |s| < 0.001$ will do.

Exercise 11–2. Since $|s^5/s| = s^4$, the requirement on s is $0 < |s| < 0.1$.

§12. Seven Rules for Computing Derivatives

There are five basic ways in which two functions can be combined to form a new function: addition, subtraction, multiplication, division, and composition. The following rules allow the derivative of the function formed in any combination of these five ways to be computed in terms of the derivatives of the functions that were combined. This greatly simplifies the computation of derivatives.

Addition Rule for Derivatives. *If $f(x)$ and $g(x)$ are functions and if $f'(x)$ and $g'(x)$ both exist and are finite then $(f+g)'(x) = f'(x) + g'(x)$.*

proof : The objective is to compute $(f+g)(x+s)$ for small s and identify the term of order s. Since $f(x+s) = f(x) + sf'(x) + o(s)$ and $g(x+s) = g(x) + sg'(x) + o(s)$, simple computation gives

$$\begin{aligned}(f+g)(x+s) &= f(x+s) + g(x+s) \\ &= f(x) + sf'(x) + g(x) + sg'(x) + o(s) \\ &= (f+g)(x) + s(f'(x) + g'(x)) + o(s).\end{aligned}$$

Hence $(f+g)'(x) = f'(x) + g'(x)$, as claimed. ∎

Example 12–1. Using this rule, $(x + x^2)' = (x)' + (x^2)' = 1 + 2x$.

Example 12–2. An earlier problem established the **Constant Multiple Rule for Derivatives**: if c is a number and if $f'(x)$ exists and is finite then $(cf(x))' = cf'(x)$. Using this and the addition rule gives $(3x^2 + 5x)' = 6x + 5$.

Subtraction Rule for Derivatives. *If $f(x)$ and $g(x)$ are functions and if $f'(x)$ and $g'(x)$ both exist and are finite then $(f-g)'(x) = f'(x) - g'(x)$.*

Exercise 12–1. Prove the Subtraction Rule for Derivatives.

Multiplication Rule for Derivatives (Product Rule). *If $f(x)$ and $g(x)$ are functions and if $f'(x)$ and $g'(x)$ both exist and are finite then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.*

proof : The technique is much the same as that used above.

$$\begin{aligned}(fg)(x+s) &= f(x+s)g(x+s) \\ &= (f(x) + sf'(x) + o(s))(g(x) + sg'(x) + o(s)) \\ &= f(x)g(x) + sf'(x)g(x) + sf(x)g'(x) + o(s) \\ &= (fg)(x) + s(f'(x)g(x) + f(x)g'(x)) + o(s).\end{aligned}$$

Thus $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. ∎

Example 12–3. Using the product rule, $(x^3)' = (xx^2)' = 1x^2 + x(2x) = 3x^2$. Similarly, $(x^4)' = (xx^3)' = 1x^3 + x(3x^2) = 4x^3$. Continuing in this way gives the **Power Rule for Derivatives**: $(x^n)' = nx^{n-1}$. Notice that this proof is only valid for positive integer powers n. The power rule is possibly the most often used rule for computing derivatives.

Example 12–4. Using the power rule and the addition and subtraction rules gives $(4x^3 - 7x^2 + 9)' = 12x^2 - 14x$.

Division Rule for Derivatives (Quotient Rule). *If $f(x)$ and $g(x)$ are functions and if $f'(x)$ and $g'(x)$ both exist and are finite and if $g(x) \neq 0$ then $(f/g)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.*

proof : Once again, the method is similar to the preceding.

$$\frac{f(x+s)}{g(x+s)} = \frac{f(x) + sf'(x) + o(s)}{g(x) + sg'(x) + o(s)}.$$

To continue, perform long division in order to simplify the quotient. This gives

$$\frac{f(x) + sf'(x) + o(s)}{g(x) + sg'(x) + o(s)} = \frac{f(x)}{g(x)} + s\frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} + o(s).$$

Once again, the rule is proved. ∎

Example 12–5. Using the quotient rule and the other rules gives $\dfrac{d}{dx}\dfrac{2x-7}{x^2+12x} = \dfrac{(x^2+12x)2 - (2x-7)(2x+12)}{(x^2+12)^2}$.

Example 12–6. Using the quotient rule and the power rule shows that $(x^{-n})' = (1/x^n)' = -nx^{n-1}/x^{2n} = -nx^{-n-1}$ for positive n. Thus the power rule also works for negative powers.

Composition Rule for Derivatives (Chain Rule). *If $f(x)$ and $g(x)$ are functions and if $f'(g(x))$ and $g'(x)$ both exist and are finite then $(f(g(x))' = f'(g(x))g'(x)$.*

proof : Proceding as before,

$$f(g(x+s)) = f(g(x) + sg'(x) + o(s)).$$

Now $g(x)$ does not depend on s, and $sg'(x) + o(s)$ is itself small, so using these two quantities in place of x and s in the defining equation for the derivative of f gives

$$\begin{aligned} f(g(x) + sg'(x) + o(s)) &= f(g(x)) + (sg'(x) + o(s))f'(g(x)) + o(s) \\ &= f(g(x)) + sf'(g(x))g'(x) + o(s) \end{aligned}$$

which completes the proof. ∎

Example 12–7. Using the chain rule makes computing the derivative of $(x^2 + 7x + 1)^{100}$ simple. In this case, taking $f(x) = x^{100}$ and $g(x) = x^2 + 7x + 1$ gives

$f(g(x)) = (x^2 + 7x + 1)^{100}$. Now $f'(x) = 100x^{99}$ by the power rule, and $g'(x) = 2x + 7$. So finally, $\frac{d}{dx}(x^2 + 7x + 1)^{100} = 100(x^2 + 7x + 1)^{99}(2x + 7)$.

A certain amount of computational profiency is required in using these rules to compute derivatives. The objective should be to develop enough skill to compute derivatives needed in working out the simple examples and applications that will follow. The derivatives of complicated expression are often best computed with a computer.

Problems

Problem 12–1. If $f(t) = 3t^2 - 7t + 6$ compute $f'(t)$.

Problem 12–2. Compute $\dfrac{d}{dt}\dfrac{(t^2+1)^5}{t-1}$.

Problem 12–3. Compute $\dfrac{d}{dx}(3x^2 - 7x + 9/x)$.

Problem 12–4. If $g(w) = 2w^{-3} + 4w^2$ compute $g'(3)$.

Problem 12–5. If $h(z) = (2z+7)^4$ compute $\left.\dfrac{d}{dz}h(z)\right|_{z=-1}$.

Problem 12–6. The value $A(i)$ of portfolio A depends on the prevailing interest rate i and is given by the formula $A(i) = \dfrac{500}{(1+i)} + \dfrac{500}{(1+i)^2}$. The value $B(i)$ of portfolio B also depends on the interest rate i and is given by $B(i) = \dfrac{1000}{(1+i)^2}$. Compute the value of portfolio A when the interest rate is 5% and when the interest rate is 6%. Which is larger? The quantity $-A'(i)$ (or $-B'(i)$) is called the **dollar duration** of the portfolio. Why does the dollar duration measure the sensitivity of the value of the portfolio to changes in the prevailing interest rate? What is the significance of the minus sign in the definition of dollar duration? Which portfolio, A or B, would be preferred by an investor wishing to minimize risk due to interest rate fluctuations from the current 5% level? Why?

Solutions to Problems

Problem 12–1. $f'(t) = 6t - 7$.

Problem 12–2. $\dfrac{d}{dt}\dfrac{(t^2+1)^5}{t-1} = \dfrac{(t-1)5(t^2+1)^4(2t)-(t^2+1)^5}{(t-1)^2}$.

Problem 12–3. $\dfrac{d}{dx}(3x^2 - 7x + 9/x) = 6x - 7 - 9/x^2$.

Problem 12–4. Using the addition and power rules gives $g'(w) = -6w^{-4} + 8w$, from which $g'(3) = -6/3^4 + 24$.

Problem 12–5. Here the chain rule gives $h'(z) = 4(2z+7)^3(2)$, so that $h'(-1) = 8(5)^3 = 1000$.

Problem 12–6. Simple computation gives $A(0.05) = 500/(1+0.05) + 500/(1+0.05)^2 = 929.71$ and $A(0.06) = 916.70$. So $A(0.05)$ is larger. By definition, $A'(i)$ is the rate at which the value of portfolio changes as the interest rate i changes. Since increasing i decreases the value of the portfolio, the minus sign simply arranges things so that a larger duration corresponds to a larger change in value for a fixed change in interest rate. For an interest rate of 5%, the dollar durations are computed to be $-A'(0.05) = 1317.35$ and $-B'(0.05) = 1727.67$. The dollar durations show that portfolio B is more sensitive to interest rate changes and the value of portfolio B will change more greatly in value if interest rates deviate from the current level. Thus portfolio A would be preferred by such an investor.

Solutions to Exercises

Exercise 12–1. The proof is quite similar to the proof of the addition rule. The objective is to compute $(f-g)(x+s)$ for small s and identify the term of order s. Since $f(x+s) = f(x) + sf'(x) + o(s)$ and $g(x+s) = g(x) + sg'(x) + o(s)$, simple computation gives

$$\begin{aligned}(f-g)(x+s) &= f(x+s) - g(x+s)\\ &= f(x) + sf'(x) - (g(x) + sg'(x)) + o(s)\\ &= (f-g)(x) + s(f'(x) - g'(x)) + o(s).\end{aligned}$$

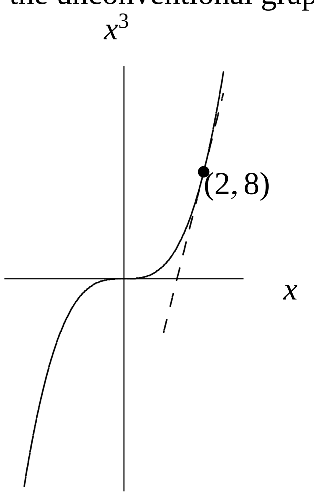
Hence $(f-g)'(x) = f'(x) - g'(x)$, as claimed.

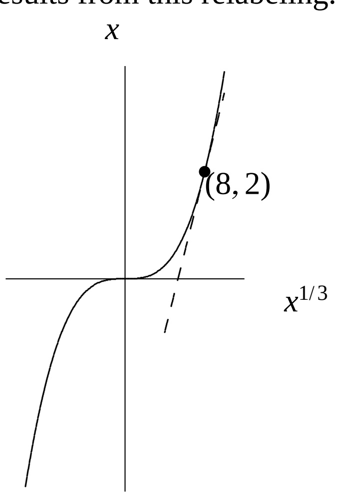
§13. Functions and Inverse Functions Revisited

The connection between the derivative of a function and the derivative of its inverse function is investigated. The immediate objective is to verify the power rule when the power is a rational number, that is, the ratio of two integers.

When a function f has an inverse function f^{-1}, the intimate relationship between f and f^{-1} should translate into a close relationship between the derivatives of these two functions. Geometric reasoning can be used to visualize this relationship.

As a prototype, consider the function $c(x) = x^3$. The function c has inverse function $c^{-1}(x) = x^{1/3}$. How are the graphs of these two functions related? For a concrete case, the point $(2, 8)$ lies on the graph of c since $2^3 = 8$. Thus $c^{-1}(8) = 2$, and the point $(8, 2)$ lies on the graph of c^{-1}. Extending this line of reasoning shows that the point (a, b) lies on the graph of c if and only if the point (b, a) lies on the graph of c^{-1}. An unconventional graph of c^{-1} can be obtained simply by relabeling the axes, starting with a conventional graph of c: relabel the positive y axis on the graph of c as the positive x axis for the graph of c^{-1}, and relabel the positive x axis on the graph of c as the positive y axis for the graph of c^{-1}. This scheme is illustrated in the following two graphs. The first is a conventional graph of $c(x) = x^3$, while the second is the unconventional graph of c^{-1} that results from this relabeling.





More information can be gleaned from the two graphs. The approximating line to the graph of c at the point $(2, 8)$ is geometrically the same line as the approximating line to the graph of c^{-1} at the point $(8, 2)$. The only difference is the coordinate system in which the line is viewed in the two cases. In the coordinate system for the graph of c, the line has slope $c'(2) = 3(2)^2 = 12$. In the coordinate system for the graph of c^{-1} the line must therefore have slope $1/12$, because of the relationship between the two coordinate systems. So $c^{-1\prime}(8) = 1/12$.

This line of geometric reasoning is completely general. The function f has a derivative at $x = a$ if and only if the inverse function f^{-1} has a derivative at $f(a)$, and ${f^{-1}}'(f(a)) = 1/f'(a)$. Since this relationship holds for any point a in the domain of f, making the substitution $a = f^{-1}(x)$ gives ${f^{-1}}'(x) = 1/f'(f^{-1}(x))$ for any point x in the domain of f^{-1}. Note that this formula can only be applied when the denominator of the right side of the equation is not zero.

To complete the prototype example, $(x^{1/3})' = 1/3(x^{1/3})^2 = (1/3)x^{-2/3}$, which agrees with the power rule for derivatives.

Exercise 13–1. What is the domain of $x^{1/3}$? What is the domain of $(x^{1/3})'$?

Example 13–1. An alternate algebraic derivation of the computational formula for ${f^{-1}}'$ can be obtained by starting with the equation $f(f^{-1}(x)) = x$, which is one of the defining relations for the inverse function. Since this equality holds for all x in the range of f, differentiation of both sides of this equality leads to the equation $f'(f^{-1}(x)){f^{-1}}'(x) = 1$ by the chain rule. Solving gives ${f^{-1}}'(x) = 1/f'(f^{-1}(x))$ as before. Notice that this derivation does not establish the *existence* of the derivative of the inverse function, but gives a formula for computing the derivative of the inverse function if the derivative exists. The geometric approach here shows that f has a derivative if and only if f^{-1} does.

This same line of reasoning can not be immediately extended to the function $f(x) = x^2$, since this function does not have an inverse function. A way around this difficulty is possible by noticing that the reason f fails to have an inverse function is that the default domain of f is too big: f maps two points in the domain to the same point in the range. A function that is closely related to f and does have an inverse function can be obtained by using the rule of f on a smaller domain. Many choices are available. Perhaps the most natural such function has domain $\{x \in \mathbf{R} : x \geq 0\}$ and the same rule as f. For convenience, write $s(x)$ for the function with this smaller domain and with rule $s(x) = x^2$. Then $s^{-1}(x) = \sqrt{x}$. The earlier argument then applies to give $\sqrt{x}' = 1/2\sqrt{x}$, that is, $(x^{1/2})' = (1/2)x^{-1/2}$. This is again in agreement with the form of the power rule. Similar reasoning shows that $(x^{1/n})' = (1/n)x^{(1/n)-1}$ for any integer n.

To conclude the extension of the power rule, write $x^{m/n} = (x^{1/n})^m$, and then use the chain rule together with this last equation to get $(x^{m/n})' = (m/n)x^{(m/n)-1}$. This is the power rule for fractional exponents.

Later on, this notion of restricting the domain of a function in order to define an inverse function will be used in other contexts.

Problems

Problem 13–1. If $f(x) = \sqrt{x^2 - 7x}$, compute $f'(x)$. What is the domain of f'?

Problem 13–2. If $g(x) = \sqrt{x} - x^{4/3}$ compute $g'(1)$. What is the domain of g'?

Problem 13–3. Compute $\dfrac{d}{dx}\dfrac{x^2 + 4x + 3}{\sqrt{x}}$.

Problem 13–4. If $f(x) = 3x^2 - 12x + 19$, compute $f'(x)$.

Problem 13–5. If $g(y) = (y^4 - 2)^5$, compute $g'(1)$.

Problem 13–6. If h has an inverse function, and if $h(3) = 12$ and $h'(3) = -5$, compute $h^{-1\prime}(12)$.

Problem 13–7. Suppose $h(x)$ is a function with the property that $h'(x) = 1/x$. How are h^{-1} and $h^{-1\prime}$ related?

Problem 13–8. Let $S(t)$ be the rate, in millimeters per hour, at which snow is falling at time t and let $M(t)$ be the rate, in millimeters per hour, at which snow is melting at time t. Let $A(t)$ be the depth, in millimeters, of snow on the ground at time t. What physical quantity does $\int_0^5 S(t)\,dt$ represent in the present context? Write a formula for $A(t)$ in terms of $S(t)$, $M(t)$, and $A(0)$. Write a formula for the rate at which snow is accumulating on the ground.

Solutions to Problems

Problem 13–1. Using the chain rule and power rule gives $f'(x) = (x^2 - 7x)^{-1/2}(2x-7)/2$. The domain of f' is $\{x \in \mathbf{R} : x^2 - 7x > 0\}$.

Problem 13–2. Here $g'(x) = 1/2\sqrt{(}x) - (4/3)x^{1/3}$ so $g'(1) = (1/2) - (4/3) = -5/6$. The domain of g' is $\{x \in \mathbf{R} : x \neq 0\}$.

Problem 13–3. Instead of using the quotient rule, first divide through to obtain $\dfrac{x^2+4x+3}{\sqrt{x}} = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$, from which the derivative is $(3/2)x^{1/2} + 2x^{-1/2} - (3/2)x^{-3/2}$, by the power rule.

Problem 13–4. $f'(x) = 6x - 12$.

Problem 13–5. $g'(y) = 5(y^4-2)^4 \times 4y^3$, by the chain rule and power rule. So $g'(1) = 20$.

Problem 13–6. $h^{-1\prime}(12) = -1/5$.

Problem 13–7. From the discussion in this section, $h^{-1\prime}(x) = 1/h'(h^{-1}(x)) = h^{-1}(x)$, using the fact that $h'(x) = 1/x$. Thus h^{-1} is a function which is its own derivative.

Problem 13–8. This integral $\int_0^5 S(t)\,dt$ is the amount of snow, in millimeters, that fell between time 0 and 5. $A(t) = A(0) + \int_0^t S(t) - M(t)\,dt$. The rate of accumulation is $A'(t) = S(t) - M(t)$.

Solutions to Exercises

Exercise 13–1. The domain of $x^{1/3}$ is all real numbers, while $(x^{1/3})'$ has domain all non-zero real numbers.

§14. Seven Rules for Computing Integrals

Each rule for computing derivatives has a parallel rule for computing integrals. The bridge between these parallel sets of rules is the Fundamental Theorem of Calculus.

The Fundamental Theorem states that $\int_a^b f'(t)\,dt = f(b) - f(a)$. In fact, more is true. If g is *any* function with $g'(t) = f'(t)$ for all $a \le t \le b$ then $\int_a^b f'(t)\,dt = g(b)-g(a)$. In practical terms, this means that evaluating an integral involves finding *any* function whose derivative is the integrand.

Example 14–1. In order to compute $\int_0^2 t^3\,dt$ the first objective is to find a function whose derivative is t^3. Since by the power rule for derivatives $\frac{d}{dt}t^4 = 4t^3$, the function $t^4/4$ has the desired property. Using the Fundamental Theorem gives $\int_0^2 t^3\,dt = 2^4/4 - 0^4/4$.

The methodology of the example can be generalized to handle any power.

Power Rule for Integrals. *For any $n \neq -1$ and any $a < b$ with the property that the interval $a \le t \le b$ lies entirely in the domain of t^n, $\int_a^b t^n\,dt = \dfrac{b^{n+1}}{(n+1)} - \dfrac{a^{n+1}}{(n+1)}$.*

Exercise 14–1. Prove the Power Rule for Integrals.

Exercise 14–2. Why is $n = -1$ excluded from the Power Rule for Integrals?

Example 14–2. The constant multiple rule for derivatives states that for any number c, $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$. The Fundamental Theorem then gives the **Constant Multiple Rule for Integrals** as $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

The other rules developed earlier for computing derivatives can be turned into rules for computing integrals in the same methodical way.

Addition Rule for Integrals. *If h and j are functions defined on the interval $a \le x \le b$ and if $\int_a^b h(x)\,dx$ and $\int_a^b j(x)\,dx$ are both finite then $\int_a^b h(x) + j(x)\,dx = \int_a^b h(x)\,dx + \int_a^b j(x)\,dx$.*

proof : If H is any function for which $H'(x) = h(x)$ for $a \le x \le b$ and if J is any function for which $J'(x) = j(x)$ for $a \le x \le b$ then $H + J$ is a function for which $(H + J)'(x) = H'(x) + J'(x)$ for

$a \le x \le b$, by the addition rule for derivatives. Thus by the Fundamental Theorem,

$$\begin{aligned}\int_a^b h(x) + j(x)\,dx &= (H+J)(b) - (H+J)(a) \\ &= H(b) - H(a) + J(b) - J(a) \\ &= \int_a^b h(x)\,dx + \int_a^b j(x)\,dx\end{aligned}$$

which is what was to be proved. ∎

The subtraction rule for derivatives also has an integral analog.

Subtraction Rule for Integrals. *If h and j are functions defined on the interval $a \le x \le b$ and if $\int_a^b h(x)\,dx$ and $\int_a^b j(x)\,dx$ are both finite then $\int_a^b h(x) - j(x)\,dx = \int_a^b h(x)\,dx - \int_a^b j(x)\,dx$.*

Exercise 14–3. Prove the Subtraction Rule for Integrals.

Example 14–3. Using the rules developed thus far gives $\int_0^4 3t^2 - 2t + 7\,dt = \int_0^4 3t^2\,dt - \int_0^4 2t + \int_0^4 7\,dt = 4^3 - 4^2 + 7(4)$.

The product rule for derivatives has a useful integral analog.

Integration by Parts. *If h and j are functions defined on the interval $a \le x \le b$ and if $\int_a^b h(x)j'(x)\,dx$ and $\int_a^b h'(x)j(x)\,dx$ exist and are finite then $\int_a^b h(x)j'(x)\,dx = h(b)j(b) - h(a)j(a) - \int_a^b h'(x)j(x)\,dx$.*

proof : The product rule for derivatives gives $(hj)'(x) = h(x)j'(x) + h'(x)j(x)$. Integrating both sides of this equation and applying the Fundamental Theorem yields $h(b)j(b) - h(a)j(a) = \int_a^b (hj)'(x)\,dx = \int_a^b h(x)j'(x)\,dx + \int_a^b h'(x)j(x)\,dx$, from which the rule follows by rearranging terms. ∎

The quotient rule for derivatives does not yield an integration rule that has proved to be useful. The chain rule provides two of the most useful integration formulas. These formulas should be thought of as formulas that are useful to rewrite complicated integrals in a simpler form, so that the other integration rules can be more easily used to complete the computation.

Substitution Rule for Integrals. *If g has a derivative for each point $a < x < b$ and f has a derivative at each point $g(a) < x < g(b)$, then $\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(x)\,dx$.*

Exercise 14–4. Prove the Substitution Rule for Integrals.

Example 14–4. A complicated appearing integral such as $\int_0^1 (x+12)^9\,dx$ can be reduced to a simpler form. Here take $f'(x) = x^9$ and $g(x) = x + 12$. Since $g'(x) = 1$, the substitution rule gives $\int_0^1 (x+12)^9\,dx = \int_{12}^{13} x^9\,dx$. The power rule can be applied to complete the computation.

Exercise 14–5. Complete the computation.

Example 14–5. The methodology can be extended a bit. To compute $\int_0^1 (2x + 12)^9\,dx$, take $f'(x) = x^9$ and $g(x) = 2x + 12$. Then $g'(x) = 2$, and this extra 2 does not appear in the integral. But the invisible 1 factor in the integrand can be written as $2(1/2)$ thus making a 2 appear. The substitution rule then gives $\int_0^1 (2x+12)^9\,dx = \int_0^1 (1/2)2(2x+12)^9\,dx = \int_{12}^{13} (1/2)x^9\,dx = (1/2)\int_{12}^{13} x^9\,dx$. The power rule can now be used to complete the computation.

Example 14–6. The integral $\int_0^1 x(x+12)^{12}\,dx$ could be computed by expanding out the integrand and using the power rule. If the leading x were absent, the substitution rule could be applied as in the preceding examples. The integration by parts formula can be used to make the leading x disappear. Take $h(x) = x$ and $j'(x) = (x+12)^{12}$ and use the integration by parts rule to obtain $\int_0^1 x(x+12)^{12}\,dx = 1(13)^{13}/13 - \int_0^1 (x+12)^{13}/13\,dx$. The last integral can now be evaluated by substitution.

Exercise 14–6. Compute $\int_0^1 (x+12)^{13}\,dx$.

The substitution rule makes no requirements about whether the function g has an inverse function or not. If g (with domain $a \le x \le b$) does have an inverse function the substitution rule can be rewritten in a slightly different form.

Change of Variable Rule for Integrals. *If g^{-1} exists and has a derivative at each point $a < x < b$ then* $\int_a^b f'(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f'(g(x))g'(x)\,dx.$

proof : Start with the substitution rule and replace a with $g^{-1}(a)$ and b with $g^{-1}(b)$. ∎

At first glance, the change of variable rule seems to be taking a simpler looking integral on the left and replacing it with a more complicated integral on the right. Amazingly, the more complicated expression in the theorem will actually appear simpler in most problems in which the change of variable is applied. The key observation is that g can be chosen arbitrarily, subject only to the condition that g^{-1}, with domain $a \le x \le b$, has an inverse function.

Example 14–7. Here's an amusing illustration of the change of variable theorem. Take $f'(x) = 1/x$ and $g(x) = 2x$ and apply the theorem to obtain $\int_2^8 1/x\,dx = \int_1^4 1/x\,dx$. You might wish to study the graph of $1/x$ and see if you think this equality of areas is obvious or amazing.

Problems

Problem 14–1. Compute $\int_0^1 \sqrt{1-x^2}\,dx$ by giving the integral a geometric interpretation as an area.

Problem 14–2. If $g(t) = \dfrac{t}{t^2+1}$ then $g'(t) =$

Problem 14–3. $\displaystyle\int_0^1 \frac{(t^2+1) - t(2t)}{(t^2+1)^2}\,dt =$

Problem 14–4. Compute $\int_0^1 x^{7/4}\,dx$.

Problem 14–5. What is $\int_{-2}^3 t^{-3}\,dt$?

Problem 14–6. Compute $\int_0^3 x(3+x^4)\,dx$.

Problem 14–7. Use the change of variables formula to show that $\int_0^1 \sqrt{1-t^2}\,dt = \int_0^1 2t\sqrt{1-t^4}\,dt$.

Problem 14–8. Compute $\int_0^5 \sqrt{4+9z}\,dz$.

Problem 14–9. Compute $\int_0^2 w(1+3w^2)^8\,dw$.

Problem 14–10. Compute $\int_{-2}^2 (t+3)\sqrt{4-t^2}\,dt$.

Problem 14–11. The ideal gas law states that the pressure $P(t)$, volume $V(t)$, and temperature $T(t)$ of a gas at time t satisfy the relationship $P(t)V(t) = T(t)$. A gas undergoes a change in pressure and volume which causes the pressure to increase at a constant rate of 20 kilopascals per second while the volume increases at a constant rate of 30 liters per second. At what rate is the temperature changing when the temperature is 100 and the pressure is 40 kilopascals?

Problem 14–12. Refer to the previous problem. In an air conditioning exchange unit the coolant undergoes an instantaneous decompression from a pressure of 1400 kilopascals to 100 kilopascals while the volume increases instantaneously from 0.1 liter to 1 liter. What temperature change results? Hint: Don’t be too quick to differentiate. What does ‘instantaneous’ mean?

Solutions to Problems

Problem 14–1. The integral is the area of a quarter of a circle of radius 1 centered at the origin. The value of the integral is thus $\pi/4$.

Problem 14–2. Using the quotient rule, $g'(t) = \dfrac{(t^2+1) - t(2t)}{(t^2+1)^2} = \dfrac{1-t^2}{(1+t^2)^2}$.

Problem 14–3. By the previous problem, $\displaystyle\int_0^1 \frac{(t^2+1) - t(2t)}{(t^2+1)^2}\,dt = \frac{1}{1^2+1} - \frac{0}{0^2+1} = 1/2$.

Problem 14–4. Using the power rule gives $\displaystyle\int_0^1 x^{7/4}\,dx = 4/11$.

Problem 14–5. This integral does not exist, since the domain of the function t^{-3} excludes zero, but zero lies in the interval of integration.

Problem 14–6. Writing $x(3+x^4) = 3x + x^5$ and using the power rule gives the value as $27/2 + 729/6$.

Problem 14–7. Try $g(t) = t^2$. What is g^{-1}?

Problem 14–8. Apply the substitution rule with $g(z) = 4 + 9z$ to obtain $\displaystyle\int_0^5 \sqrt{4+9z}\,dz = (1/9)\int_4^{49} \sqrt{z}\,dz$. Now use the power rule.

Problem 14–9. Apply the substitution rule with $g(w) = 1 + 3w^2$ to obtain $\displaystyle\int_0^2 w(1+3w^2)^8\,dw = (1/6)\int_1^{13} w^8\,dw$ and use the power rule.

Problem 14–10. By the addition rule $\displaystyle\int_{-2}^2 (t+3)\sqrt{4-t^2}\,dt = \int_{-2}^2 t\sqrt{4-t^2}\,dt + \int_{-2}^2 \sqrt{4-t^2}\,dt$. The substitution rule with $g(t) = 4 - t^2$ will allow the evaluation of the first integral. Alternately, interpreting the first integral geometrically shows that it is zero. The second integral has a simple geometric interpretation also, which gives its value as 2π.

Problem 14–11. Differentiating the relationship of the ideal gas law gives $T'(t) = P(t)V'(t) + P'(t)V(t)$, by the product rule. At the time t_0 in question $P'(t_0) = 20$ and $V'(t_0) = 30$ while $T(t_0) = 100$ and $P(t_0) = 40$. From the ideal gas law these last two pieces of information give $V(t_0) = 100/40$. Hence $T'(t_0) = 40 \times 30 + 20 \times (100/40) = 1250$.

Problem 14–12. The fact that the change is instantaneous means that neither $P(t)$ nor $V(t)$ have a derivative at the time at which the change occurs. Just prior to the change the temperature is $1400 \times 0.1 = 140$ while just after the change the temperature is $100 \times 1 = 100$. So the temperature change is –40.

Solutions to Exercises

Exercise 14–1. By the power rule for derivatives, $\frac{d}{dt}t^{n+1}/(n+1) = t^n$, provided $n \neq -1$. The result follows from the Fundamental Theorem.

Exercise 14–2. When $n = -1$, $t^{n+1} = t^0 = 1$ and $\frac{d}{dt}t^0 = 0$ which is not t^{-1}.

Exercise 14–3. If H is any function for which $H'(x) = h(x)$ for $a \le x \le b$ and if J is any function for which $J'(x) = j(x)$ for $a \le x \le b$ then $H - J$ is a function for which $(H - J)'(x) = H'(x) - J'(x)$ for $a \le x \le b$, by the subtraction rule for derivatives. Thus by the Fundamental Theorem,

$$\begin{aligned}\int_a^b h(x) - j(x)\,dx &= (H-J)(b) - (H-J)(a)\\ &= H(b) - H(a) - (J(b) - J(a))\\ &= \int_a^b h(x)\,dx - \int_a^b j(x)\,dx\end{aligned}$$

as desired.

Exercise 14–4. The chain rule for derivatives gives $(f \circ g)'(x) = f'(g(x))g'(x)$. Applying the Fundamental Theorem twice gives $\int_a^b f'(g(x))g'(x)\,dx = f(g(b)) - f(g(a)) = \int_{g(a)}^{g(b)} f'(x)\,dx$, as desired.

Exercise 14–5. $\int_0^1 (x+12)^9\,dx = \int_{12}^{13} x^9\,dx = (13^{10} - 12^{10})/10$, by the power rule.

Exercise 14–6. Letting $g(x) = x + 12$ and using the substitution rule gives $\int_0^1 (x+12)^{13}\,dx = \int_{12}^{13} x^13\,dx = 13^{14}/14 - 12^{14}/14$, by the power rule.

§15. Extensions and Applications

The next several sections examine some extensions and applications of the rules for computing derivatives and integrals. The extensions include the definition of a new function, the natural logarithm, in terms of integrals, the development of the exponential function, and calculus properties of the trigonometric functions. These extensions are used to solve some problems that have been previously encountered, and also to study the problem of optimization.

§16. The Logarithm and Exponential Functions

New functions can be defined in terms of old ones by using calculus ideas. Here the natural logarithm function is defined in terms of an integral. The inverse function of the natural logarithm function is the exponential function, which is the most important function in applied mathematics.

To plug the gap in the power rule for integrals, define a new function, the **natural logarithm**, by the formula $\ln(x) = \int_1^x \frac{1}{t}\,dt$. The domain of this function is $\{x \in \mathbf{R} : x > 0\}$. Since for $0 < x < 1$, $\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$, the values of $\ln(x)$ are negative for $0 < x < 1$. The geometric reasoning that led to the Fundamental Theorem immediately gives $\frac{d}{dx}\ln(x) = 1/x$, for $x > 0$.

Exercise 16–1. What is the value of $\ln(1)$?

Exercise 16–2. Use geometric reasoning to show that $\frac{d}{dx}\ln(x) = 1/x$ for $x > 0$.

The justification of the logarithm name is that the natural logarithm turns multiplication of numbers into addition of their logarithms. As a prototype for the proof of this general fact, the equality $\ln(3 \times 5) = \ln(3) + \ln(5)$ will be established. Reasoning geometrically,

$$\begin{aligned}
\ln(3 \times 5) &= \int_1^{15} \frac{1}{t}\,dt \\
&= \int_1^{3} \frac{1}{t}\,dt + \int_3^{15} \frac{1}{t}\,dt \\
&= \ln(3) + \int_3^{15} \frac{1}{t}\,dt \\
&= \ln(3) + \int_1^{5} \frac{1}{3t} 3\,dt \\
&= \ln(3) + \int_1^{5} \frac{1}{t}\,dt \\
&= \ln(3) + \ln(5)
\end{aligned}$$

where the fourth equality makes use of the change of variables theorem with $g(t) = 3t$.

The general fact is that for any numbers $a > 0$ and $b > 0$, $\ln(ab) = \ln(a) + \ln(b)$.

When both a and b are larger than 1, the prototype proof becomes

$$\begin{aligned}\ln(ab) &= \int_1^{ab} \frac{1}{t}\,dt\\ &= \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt\\ &= \ln(a) + \int_a^{ab} \frac{1}{t}\,dt\\ &= \ln(a) + \int_1^{b} \frac{1}{at}a\,dt\\ &= \ln(a) + \int_1^{b} \frac{1}{t}\,dt\\ &= \ln(a) + \ln(b)\end{aligned}$$

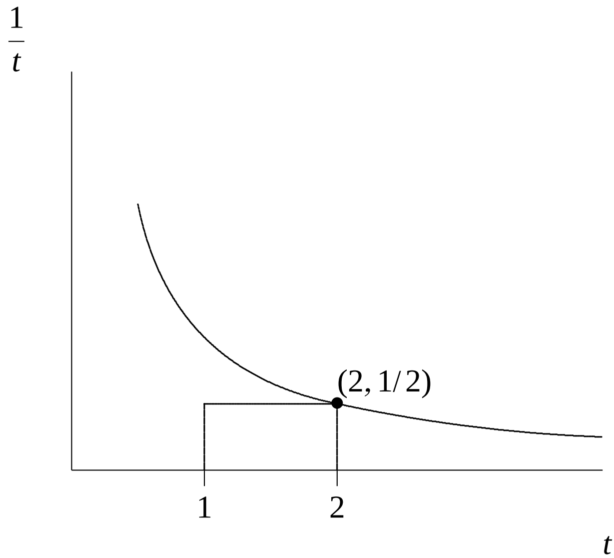
where the fourth equality makes use of the change of variables theorem with $g(t) = at$.

Exercise 16–3. How does the proof proceed if $a < ab < 1 < b$?

A second property of logarithms is that for any number $a > 0$ and any rational p, $\ln(a^p) = p\ln(a)$.

Exercise 16–4. Give a proof of this property. Hint: Use the change of variable formula with $g(t) = t^p$.

The domain of the logarithm function is the positive real numbers. What is the range of the function? The geometric considerations from the picture below show that $\ln(2)$ is larger than the area of the rectangle. Thus $\ln(2) > 1/2$.



With this information and the power property for logarithms, $\ln(2^k) = k\ln(2) > k/2$. So as x becomes arbitrarily large and positive, $\ln(x)$ does too. Similarly, $\ln(1/2) =$

$-\ln(2) < -1/2$, so $\ln((1/2)^k) < -k/2$. Thus as x approaches zero from the right, $\ln(x)$ becomes arbitrarily large and negative. The range of the function $\ln(x)$ is all real numbers.

The natural logarithm function is increasing everywhere on its domain, since $\ln(x)' = 1/x > 0$. The inverse function of the natural logarithm is called the **exponential function** and is denoted by $\exp(x)$. Since $\ln(1) = 0$, $\exp(0) = 1$. Other properties of the exponential function follow from the corresponding properties of logarithm.

For example, since $\ln(ab) = \ln(a) + \ln(b)$, the exponential function satisfies $\exp(a + b) = \exp(a)\exp(b)$. To see this, replace a with $\exp(a)$ and b with $\exp(b)$ in the logarithm relation to obtain $\ln(\exp(a)\exp(b)) = a + b$. Now exponentiate both sides to obtain $\exp(a)\exp(b) = \exp(a + b)$. Because of this fact, writing e for the number $e = \exp(1)$ provides an alternate notation $e^x = \exp(x)$.

Perhaps the most important property of the exponential function is that $\frac{d}{dx}e^x = e^x$. This follows from the general relationship between the derivative of a function and its inverse function.

Exercise 16–5. Show that $\frac{d}{dx}e^x = e^x$.

The exponential function is used to define non-rational powers. As was seen earlier, $\ln(a^p) = p\ln(a)$ for rational powers p. This equality is taken as the *definition* of $\ln(a^p)$ when p is not rational. Exponentiation of both sides of this equality gives $a^p = e^{p\ln(a)}$ as an equivalent form of this definition.

Example 16–1. The logarithm and exponential functions arise in the solution of some of the differential equations found earlier. In the earlier example of a falling ball, the velocity of the ball was found to satisfy the equation $v'(t) = g - (2/5)v(t)$ for $t > 0$ with $v(0) = 0$. An explicit expression for $v(t)$ can be found by using the **separation of variables method**. The first step of the separation of variables method is to divide both sides of the equation by the side not containing the derivative. Secondly, both sides of the resulting equation are then integrated between two convenient points. In the present case this leads to

$$\int_0^w \frac{v'(t)}{g - (2/5)v(t)}\,dt = \int_0^w 1\,dt.$$

Here the endpoints were chosen as the time $t = 0$ at which the ball was dropped and the time $t = w > 0$ representing an arbitrary future time at which the velocity is to be computed. Computing the integrals gives $(-5/2)\ln(1 - (2/5g)v(w)) = w$, and solving yields $v(w) = (5g/2)\left(1 - e^{-(2/5)w}\right)$ for $w \geq 0$.

Exercise 16–6. Fill in the details of the computation.

Problems

Problem 16–1. Define $g(x) = \int_1^x \frac{1}{t^2}\,dt$, for $x > 0$. What is the range of g? For $x > 1$, which is larger $g(x)$ or $\ln(x)$?

Problem 16–2. Compute $\frac{d}{dx} \ln|x|$.

Problem 16–3. If p is not rational, compute $\frac{d}{dx}x^p$ in order to complete the extension of the power rule.

Problem 16–4. Compute $\frac{d}{dx}2^x$.

Problem 16–5. Compute $\int_0^2 e^{5x}\,dx$.

Problem 16–6. True or False: $\int_0^{10^{23}} e^{-t^2}\,dt > 0$.

Problem 16–7. Compute $\frac{d}{dz}\int_z^{z^2} e^{w^2}\,dw$.

Problem 16–8. Newton's Law of Cooling states that the rate of cooling of a body immersed in a bath is proportional to the difference in temperature between the body and the bath. Let $T(t)$ denote the temperature of the body at time t, B denote the (constant) temperature of the bath, and assume the proportionality constant is 5. Write an equation involving T and B that expresses Newton's Law of Cooling. Make sure your equation gives the rate of cooling the proper sign. If $B = 72$ and $T(1) = 80$, what was $T(0)$?

Problem 16–9. Torricelli's Law states that the rate at which fluid leaves a container through a small hole in the bottom of the container is proportional to the square root of the depth of fluid in the container. Let $V(t)$ denote the volume of fluid in the container at time t, $D(t)$ denote the depth of fluid in the container at time t, and assume the magnitude of the proportionality constant is 17. Write an equation involving V and D that expresses Torricelli's Law. Make sure that your equation gives the rate of change in volume the proper sign. If the container is cylindrical with base radius 2, find an expression for $V(t)$ assuming that $V(0) = 100$.

Solutions to Problems

Problem 16–1. For $x > 1$, $g(x) = 1 - 1/x$ while for $0 < x < 1$, $g(x) = -1/x + 1$. Thus the range of g is $\{x \in \mathbf{R} : -\infty < x < 1\}$. Since $1/t^2 < 1/t$ for $t > 1$, the area representing $\ln(x)$ is larger than the area representing $g(x)$. Hence $g(x) < \ln(x)$ for $x > 1$. What happens for $x < 1$?

Problem 16–2. For $x > 0$, $\dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln(x) = 1/x$, while for $x < 0$, $\dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln(-x) = \dfrac{1}{-x}(-1) = 1/x$ by the chain rule. Thus $\frac{d}{dx}\ln|x| = 1/x$ for all $x \neq 0$.

Problem 16–3. Using the definition of x^p gives $\dfrac{d}{dx}x^p = \dfrac{d}{dx}e^{p\ln(x)} = e^{p\ln(x)}p/x = px^p/x = px^{p-1}$. For which combinations of x and p is this formula valid?

Problem 16–4. Since $2^x = e^{x\ln(2)}$, the derivative is $2^x\ln(2)$.

Problem 16–5. Using the substitution rule with $g(x) = 5x$ gives $\displaystyle\int_0^2 e^{5x}\,dx = (1/5)\int_0^{10} e^x\,dx = (e^{10} - 1)/5$, by the Fundamental Theorem.

Problem 16–6. True, since the exponential function always has positive values, the graph of the integrand lies entirely above the axis.

Problem 16–7. If $f(x) = \int_0^x e^{w^2}\,dw$, then the function to be differentiated is $f(z^2) - f(z)$. Now use the chain rule and the fact that $f'(x) = e^{x^2}$.

Problem 16–8. Newton's cooling law is $T'(t) = 5(B - T(t))$. The separation of variables method, integrating from $t = 0$ to $t = 1$, gives $-\ln\left(\dfrac{B - T(1)}{B - T(0)}\right) = 5$. Plugging in and solving gives $T(0) = 72 + 8e^5 \approx 1259.30$.

Problem 16–9. Torricelli's law is $V'(t) = -17\sqrt{D(t)}$. For the cylindrical container, $V(t) = 4\pi D(t)$, so $V'(t) = -17\sqrt{V(t)/4\pi}$. Separation of variables, integrating from $t = 0$ to $t = w > 0$, gives $2(\sqrt{V(w)} - \sqrt{V(0)}) = -17w/\sqrt{4\pi}$. Thus $V(w) = \left(10 - 17w/2\sqrt{4\pi}\right)^2$. Notice that this formula for V is only valid for $0 \le w \le 20\sqrt{4\pi}/17$. Why?

Solutions to Exercises

Exercise 16–1. From the definition, $\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$, since there is zero area enclosed by the graph.

Exercise 16–2. For small s, $\ln(x+s) = \int_1^{x+s} \frac{1}{t}\,dt = \int_1^x \frac{1}{t}\,dt + \int_x^{x+s} \frac{1}{t}\,dt = \ln(x) + s/x + o(s)$, since the area between the graph of $1/t$ and the horizontal axis for $x \le t \le x+s$ is approximately the area of a rectangle with side lengths $1/x$ and s. The error of this rectangular approximation is at most $(1/x - 1/(x+s))s = s^2/(x(x+s)) = o(s)$.

Exercise 16–3. In this case, $\int_a^b \frac{1}{t}\,dt = \int_a^{ab} \frac{1}{t}\,dt + \int_{ab}^1 \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt$. The integral on the left of the equality is $\int_a^1 \frac{1}{t}\,dt + \int_1^b \frac{1}{t}\,dt = -\ln(a) + \ln(b)$. The 3 integrals on the right are $\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt = \ln(b)$ (by the change of variables formula), $-\ln(ab)$, and $\ln(b)$. Thus $-\ln(a) + \ln(b) = \ln(b) - \ln(ab) + \ln(b)$, from which the result follows after cancellation and rearrangement. The other cases follow by similar arguments.

Exercise 16–4. Here $\int_1^{a^p} \frac{1}{t}\,dt = \int_1^a \frac{1}{t^p} pt^{p-1}\,dt = p\int_1^a \frac{1}{t}\,dt = p\ln(a)$. Close inspection reveals that this proof only works for $a > 1$. Can you provide a proof for $0 < a < 1$? Notice that in any case the result only holds for rational numbers p.

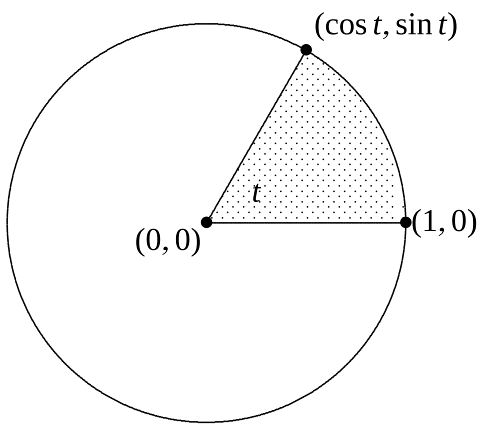
Exercise 16–5. The general relationship between the derivative of a function and the derivative of its inverse function gives $\frac{d}{dx}e^x = \frac{1}{\frac{1}{e^x}} = e^x$, since $\frac{d}{dx}\ln(x) = 1/x$.

Exercise 16–6. Here $\int_0^w 1\,dt = w$ and using the substitution rule for integrals with $f'(t) = 1/t$ and $g(t) = g - (2/5)v(t)$ gives $\int_0^w \frac{v'(t)}{g - (2/5)v(t)}\,dt = (-5/2)\int_g^{g-(2/5)v(w)} \frac{1}{t}\,dt = -(5/2)\,(\ln(g - (2/5)v(w)) - \ln(g))$. The last difference of logarithms simplifies since $\ln(a) - \ln(b) = \ln(a/b)$.

§17. Trigonometric Functions and Their Inverses

Trigonometric functions have many uses. In addition to the applications involving triangles, these functions are used to model physical phenomena which exhibit periodic behavior. The basic properties of these functions are reviewed, and their derivatives are computed.

The trigonometric functions are defined based on the **unit circle**, which is the set $\left\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 1\right\}$. The picture is as follows.



The angle measuring t radians intercepts the circle at the point which is defined to have coordinates $(\cos t, \sin t)$. Since the circle has radius 1, the Pythagorean relation

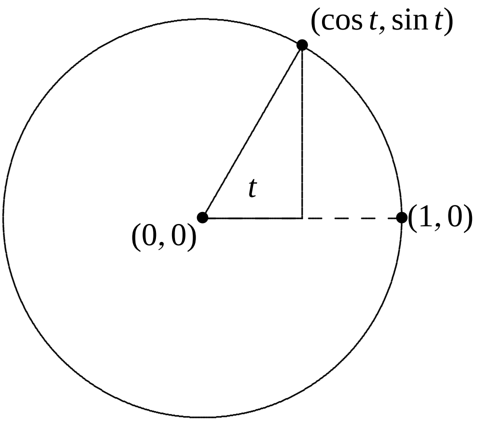
$$\cos^2 t + \sin^2 t = 1$$

is immediately seen to hold.

Recall that the radian measure of an angle is the length of the circular arc with center at the vertex of the angle that is intercepted by the angle, divided by the radius of the circular arc. In the unit circle context, the radian measure of the angle t is the same as the length of the arc of the circle intercepted by the angle. Stretching this idea a bit allows angles with arbitrary radian measure to be interpreted. If $t > 0$ is given, traverse the circumference of the unit circle in a counterclockwise direction for a distance t, starting from the point $(1, 0)$. The angle t is then identified with the terminal point of this journey. If $t < 0$, start from $(1, 0)$ and march clockwise around the circumference a distance t and identify the angle t with the point reached at the end of this journey.

This expanded notion of angle allows the domain of sine and cosine to be expanded to all real numbers. The range of the sine and cosine functions is the set $\{x \in \mathbf{R} : -1 \le x \le 1\}$.

Re-examining the previous picture with an auxillary vertical line segment added shows that the new definition of sine and cosine agrees with the definition of these functions for a right triangle, as long as the angle t lies in the interval $0 < t < \pi/2$.



Evidently neither the sine nor the cosine function has an inverse function. As with the function x^2 considered earlier, the difficulty lies in the fact that the domain has grown too big.

There are many possible ways to shrink the domain. Here is the standard method. Define a new function, also called sine, with the same rule as the usual sine function but with domain $\{x \in \mathbf{R} : -\pi/2 \leq x \leq \pi/2\}$. This special version of the sine function does have an inverse function which is denoted $\arcsin(x)$. Thus

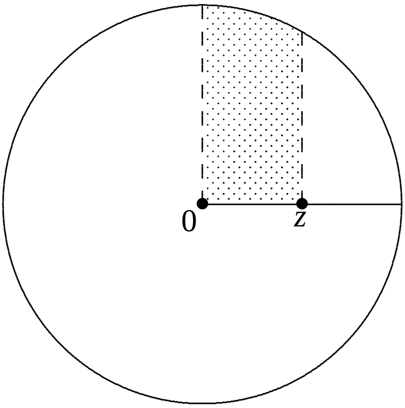
$$\sin(\arcsin(x)) = x \qquad \text{for } -1 \leq x \leq 1$$
$$\arcsin(\sin(x)) = x \qquad \text{for } -\pi/2 \leq x \leq \pi/2.$$

Similarly, define a new function, also called cosine, with the same rule as the cosine function but with domain $\{x \in \mathbf{R} : 0 \leq x \leq \pi\}$. This special version of the cosine function has an inverse function which is denoted $\arccos(x)$. Thus

$$\cos(\arccos(x)) = x \qquad \text{for } -1 \leq x \leq 1$$
$$\arccos(\cos(x)) = x \qquad \text{for } 0 \leq x \leq \pi.$$

Some authors use the notation $\sin^{-1}(x)$ and $\cos^{-1}(x)$ for $\arcsin(x)$ and $\arccos(x)$. This notation will not be used here in order to avoid confusion with powers of sine and cosine.

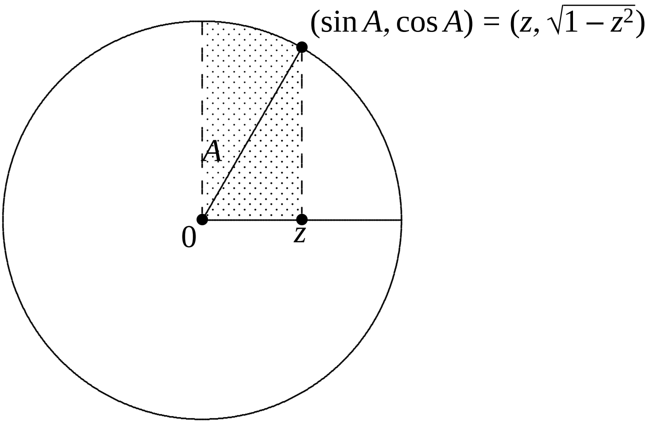
To establish some of the calculus properties of these functions, compute the area of the shaded portion of the unit circle in the picture below in two different ways.



In the upper half plane the equation of the circular arc is $y = \sqrt{1-x^2}$. Thus the area of the shaded region is

$$\int_0^z \sqrt{1-x^2}\,dx.$$

On the other hand, drawing in the auxillary line segment from the origin to unit circle decomposes the region into two pieces, a circular sector subtended by the angle A and a right triangle.



Now the dashed vertical lines are parallel, so using the alternating interior angles property shows that the point on the unit circle has coordinates $(\sin A, \cos A)$. Using the equation of the unit circle shows that the point also has coordinates $(z, \sqrt{1-z^2})$. Equating the first coordinates of these two expressions gives $\sin A = z$, so that $A = \arcsin(z)$. The circular sector has area $(\arcsin(z)/2\pi)(\pi 1^2) = \arcsin(z)/2$, since the sector is the fraction $\arcsin(z)/2\pi$ of the entire circle. The area of the right triangle is $(1/2)z\sqrt{1-z^2}$. The total area of the shaded region is $\frac{1}{2}\arcsin(z) + \frac{1}{2}z\sqrt{1-z^2}$.

Equating these two expressions for the area of the shaded region gives

$$\int_0^z \sqrt{1-x^2}\,dx = \frac{1}{2}\arcsin(z) + \frac{1}{2}z\sqrt{1-z^2}.$$

The integral has been computed by geometric reasoning.

Even more can be gleaned from the equation of the preceding paragraph. Differentiating both sides of the equation gives

$$\sqrt{1-z^2} = \frac{1}{2}\frac{d}{dz}\arcsin(z) + \frac{1}{2}z^2/\sqrt{1-z^2} + \frac{1}{2}\sqrt{1-z^2}.$$

Solving for the derivative of the arcsine function gives

$$\frac{d}{dz}\arcsin(z) = \frac{1}{\sqrt{1-z^2}}$$

which is valid for $-1 < z < 1$. Having obtained a formula for the derivative of the arcsine function, the derivative of the sine function can be found by starting with the equation

$$\arcsin(\sin(z)) = z$$

and differentiating both sides to get

$$\frac{1}{\sqrt{1-\sin^2(z)}}\frac{d}{dz}\sin(z) = 1$$

from which

$$\frac{d}{dz}\sin(z) = \sqrt{1-\sin^2(z)} = \cos(z).$$

Technically this derivation is only valid on the interval $-\pi/2 < z < \pi/2$, but because of the periodicity of the sine and cosine functions the relation holds for all z.

Similar manipulations can be used with the arccosine function. The last picture above shows that $\cos A = \sqrt{1-z^2}$ so that $A = \arccos(\sqrt{1-z^2})$. The previous argument gave $A = \arcsin z$. Equating these two expression gives $\arccos(\sqrt{1-z^2}) = \arcsin z$. Replacing z with $\sqrt{1-z^2}$ in both sides of this equality yields $\arccos z = \arcsin(\sqrt{1-z^2})$. Differentiating both sides of this equation shows that

$$\frac{d}{dz}\arccos(z) = \frac{-1}{\sqrt{1-z^2}}.$$

Also, from the equation $\arccos(\cos(z)) = z$ manipulations as above give

$$\frac{d}{dz}\cos(z) = -\sin(z)$$

thus completing the calculus picture for the trigonometric functions. Derivatives of $\tan(z) = \sin(z)/\cos(z)$ and the other trigonometric functions can then be computed using the quotient rule and the Pythagorean identity.

Problems

Problem 17–1. The tangent function is defined by $\tan(x) = \sin(x)/\cos(x)$. What is the domain and range of $\tan(x)$?

Problem 17–2. A special version of the tangent function is defined using the same rule as tangent, but with domain $-\pi/2 < x < \pi/2$. This special function has an inverse function denoted $\arctan(x)$. Compute $\frac{d}{dx}\arctan(x)$.

Problem 17–3. The secant function is defined by $\sec x = 1/\cos x$; the cosecant function is defined by $\csc x = 1/\sin x$; the cotangent function is defined by $\cot x = 1/\tan x$. Find the derivative of each of these functions.

Problem 17–4. If $f(w) = \sin(w^2)$ then $f'(w) =$

Problem 17–5. If $f(t) = t\,\sin(t)$ then $f'(t) =$

Problem 17–6. $\int_0^{\pi} t\,\cos(t) + \sin(t)\,dt =$

Problem 17–7. $\int_{-1}^{1} 2t\cos t^2\,dt =$

Problem 17–8. Compute $\int_0^1 \frac{1}{1+z^2}\,dz$.

Problem 17–9. Compute $\int_0^1 w\,\cos(w)\,dw$.

Problem 17–10. Compute $\int_0^{1/2} \frac{x}{\sqrt{1-x^4}}\,dx$.

Problem 17–11. If $g(y) = \frac{e^{y^2}}{(e^y+1)^2}$, compute $\left.\frac{d}{dy}g(y)\right|_{y=0}$.

Solutions to Problems

Problem 17–1. The domain is $\{x \in \mathbf{R} : \cos(x) \neq 0\}$, while the range is $\mathbf{R}$.

Problem 17–2. The quotient rule gives

$$\frac{d}{dx}\tan(x) = 1/\cos^2(x).$$

From $\tan(\arctan(x)) = x$, differentiation of both sides gives

$$\frac{1}{\cos^2(\arctan(x))}\frac{d}{dx}\arctan(x) = 1,$$

so that $\frac{d}{dx}\arctan(x) = \cos^2(\arctan(x))$. Dividing both sides of the Pythagorean relation by $\cos^2(x)$ gives $1 + \tan^2(x) = 1/\cos^2(x)$. Thus $\cos^2(\arctan(x)) = 1/(1 + \tan^2(\arctan(x)) = 1/(1+x^2)$. Making this substitution finally gives $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$ for $-\infty < x < \infty$.

Problem 17–3. The quotient rule gives $(\sec x)' = \sin x/\cos^2 x = \sec x \tan x$, $(\csc x)' = -\cos x/\sin^2 x = -\cot x \csc x$, and $(\cot x)' = -1/\sin^2 x = -\csc^2 x$.

Problem 17–4. $f'(w) = 2w\cos(w^2)$.

Problem 17–5. $f'(t) = t\cos(t) + \sin(t)$, by the product rule.

Problem 17–6. By the previous problem, $\int_0^{\pi} t\cos(t) + \sin(t)\,dt = \pi\sin(\pi) - 0\sin(0) = 0$.

Problem 17–7. $\int_{-1}^{1} 2t\cos t^2\,dt = \sin(1^2) - \sin((-1)^2) = 0$.

Problem 17–8. By an earlier problem, $\int_0^1 \frac{1}{1+z^2}\,dz = \arctan(1) - \arctan(0) = \pi/4$.

Problem 17–9. Integration by parts gives $\int_0^1 w\cos(w)\,dw = 1\sin(1) - \int_0^1 \sin(w)\,dw = \sin(1) + \cos(1) - 1$.

Problem 17–10. The substitution rule with $g(x) = x^2$ gives $\int_0^{1/2} \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int_0^{1/4}\frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\arcsin(1/4)$.

Problem 17–11. $g'(y) = \dfrac{(e^y+1)^2 2ye^{y^2} - e^{y^2} 2e^y(e^y+1)}{(e^y+1)^4}$, by the chain rule and quotient rule. So $g'(0) = -4/2^4 = -1/4$.

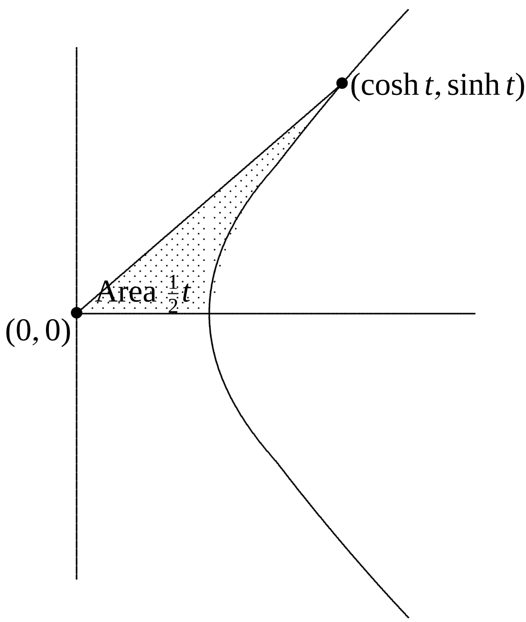
§18. Hyperbolic Trigonometry

Ordinary trigonometry is based on the unit circle. Hyperbolic trigonometry is based on the unit hyperbola $\left\{(x, y) \in \mathbf{R}^2 : x^2 - y^2 = 1\right\}$. Some basic properties of this trigonometry are now explored.

The part of the unit hyperbola lying in the right half plane is shown in the picture below. There is no obvious definition of angle in this new setting. But looking again at the unit circle shows that the angle could have been measured in terms of the area of the sector just as easily as using radian measure.

Exercise 18–1. Show that the area of the sector of the unit circle intercepted by an angle with radian measure t is $t/2$.

This observation suggests the following picture in the new case.



In this new picture, a line extends from the origin to the hyperbola $x^2 - y^2 = 1$. If the shaded area is $t/2$, the hyperbolic measure of the angle between the axis and the line is t. The point at which this line intersects the hyperbola is defined to have coordinates $(\cosh t, \sinh t)$. The two new functions are the **hyperbolic cosine function** $\cosh t$ (pronounced to rhyme with 'gosh') and the **hyperbolic sine function** $\sinh t$ (pronounced to rhyme with 'cinch'). Since the point $(\cosh t, \sinh t)$

lies on the hyperbola, the fundamental relation

$$\cosh^2 t - \sinh^2 t = 1,$$

which parallels the Pythagorean identity, is certainly valid.

Exercise 18–2. What is cosh(0)? What is sinh(0)?

What are the possible values for the hyperbolic measure of angles? The answer depends on the possible values for the enclosed area in the above picture. Since the hyperbola has slant asymptote $y = x$, the possible values for the area lie between 0 and the area between $y = x$ and the upper branch of the hyperbola. For any $w > 1$ this area exceeds $\int_1^w x - \sqrt{x^2 - 1}\,dx$. Algebra gives

$$\begin{aligned} x - \sqrt{x^2-1} &= \left(x - \sqrt{x^2-1}\right)\frac{x + \sqrt{x^2-1}}{x + \sqrt{x^2-1}} \\ &= \frac{x^2 - (x^2-1)}{x + \sqrt{x^2-1}} \\ &= \frac{1}{x + \sqrt{x^2-1}} \\ &\geq \frac{1}{2x}. \end{aligned}$$

So $\int_1^w x - \sqrt{x^2-1}\,dx \geq \int_1^w \frac{1}{2x}\,dx = \ln(w)/2$. The area can thus be any positive value. Following the custom for integration, areas below the horizontal axis are interpreted as negative numbers. Symmetry of the hyperbola about the horizontal axis shows that the enclosed area below the axis can be any negative value. So any real number can be realized as the hyperbolic measure of an angle in hyperbolic trigonometry. The domain of the hyperbolic sine and cosine functions is **R**.

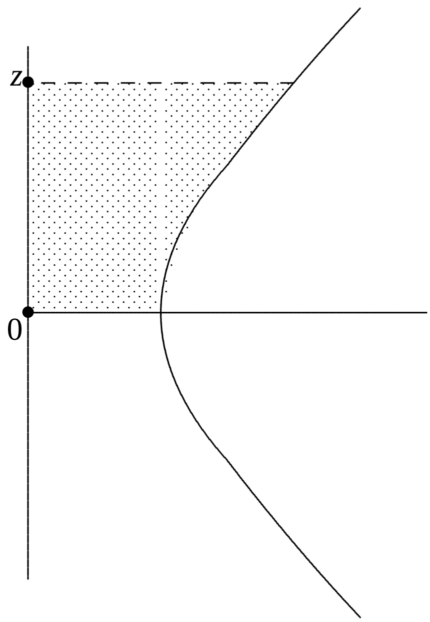
Exercise 18–3. Sketch the graph of cosh(x) and the graph of sinh(x).

By symmetry of the hyperbola about the horizontal axis, the relations $\cosh(-t) = \cosh t$ and $\sinh(-t) = -\sinh t$ must hold. The function $\sinh t$ also must have an inverse function arcsinh t whose domain is all real numbers. The equalities $\sinh(\operatorname{arcsinh} t) = t$ and $\operatorname{arcsinh}(\sinh t) = t$ hold for all real t. The hyperbolic cosine function only has an inverse function if the domain is restricted to the positive half axis. Then $\operatorname{arccosh}(\cosh(t)) = t$ for $t \geq 0$ and $\cosh(\operatorname{arccosh}(t)) = t$ for $t \geq 1$.

Exercise 18–4. Why does the domain of the hyperbolic cosine function need to be restricted in order to obtain an inverse function?

Some people use the notation $\cosh^{-1}(x)$ and $\sinh^{-1}(x)$ for the inverse functions, but this notation will not be used here.

As was the case for the usual trigonometric functions, an interesting computational formula can be derived by computing the area of the shaded region below in two ways.

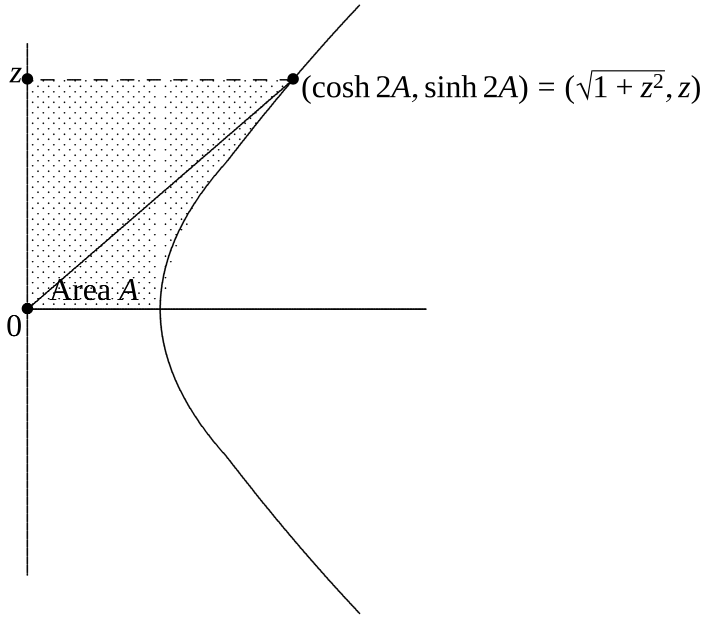


On the one hand, the area of the shaded region is

$$\int_0^z \sqrt{1+y^2}\,dy$$

since the x coordinate of points on the unit hyperbola are given by $x = \sqrt{1+y^2}$.

The shaded region is also the area of a right triangle and a triangular shaped region of area A.



Now from the definition of the hyperbolic sine function, $z = \sinh(2A)$ from which $A = \text{arcsinh}\,(z)/2$. The area of the right triangle is $(1/2)z\sqrt{1+z^2}$. The total area of the shaded region is $\frac{1}{2}\text{arcsinh}\, z + \frac{1}{2}z\sqrt{1+z^2}$.

Equating these two expressions for the area of the region gives

$$\int_0^z \sqrt{1+y^2}\, dy = \frac{1}{2}\text{arcsinh}\, z + \frac{1}{2}z\sqrt{1+z^2}$$

for all real z. Again, the integral has been computed using geometric arguments.

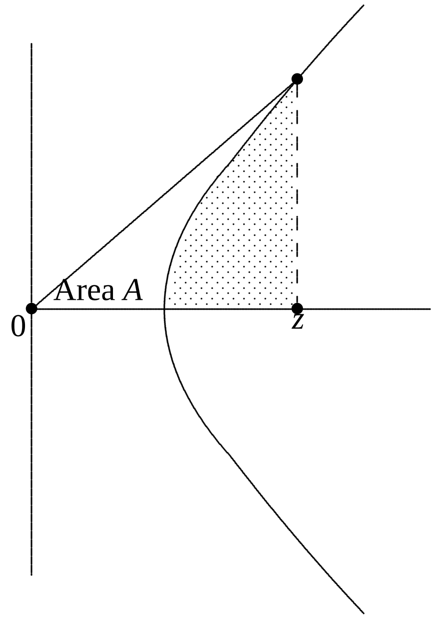
Differentiation as in the trigonometric case gives the formula

$$\frac{d}{dz}\text{arcsinh}\,(z) = \frac{1}{\sqrt{1+z^2}}$$

and using this with the relation $\text{arcsinh}\,(\sinh(z)) = z$ gives

$$\frac{d}{dz}\sinh(z) = \cosh(z).$$

Similar reasoning with the shaded region below



gives the relation

$$\frac{1}{2}\text{arccosh}\,(z) + \int_1^z \sqrt{x^2-1}\,dx = \frac{1}{2}z\sqrt{z^2-1}$$

since $\cosh(2A) = z$ from which $A = \text{arccosh}\,(z)/2$. Differentiation then gives

$$\frac{d}{dz}\text{arccosh}\,(z) = \frac{1}{\sqrt{z^2-1}},$$

for $z > 1$. Using this and the method above then gives

$$\frac{d}{dz}\cosh(z) = \sinh(z).$$

Exercise 18–5. Fill in the details in the derivation of this last equality.

These formulas for the derivatives lead to alternate expressions for computing the hyperbolic trigonometric functions. The function $\cosh x + \sinh x$ is its own derivative and takes the value 1 when x is 0. Thus $\cosh x + \sinh x = e^x$. Similarly, $\cosh x - \sinh x = e^{-x}$. Solving these two equations yields

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

as alternate computational formulas. Using these, alternate computational formulas for the inverse functions can be found. Simple algebra gives

$$\operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1})$$
$$\operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1})$$

which completes the calculus for the hyperbolic trigonometric functions.

The fact that the hyperbolic trigonometric functions are related closely to the exponential function raises the question of whether a similar relation holds for the ordinary trigonometric functions. The mathematician Leonhard Euler was one of the first to realize that if $i = \sqrt{-1}$ then $e^{ix} = \cos(x) + i\sin(x)$. This equation is usually called **Euler's identity**. Thus

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} = \cosh(ix)$$
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} = \sinh(ix)/i.$$

All of the trigonometric functions are in fact close relatives of the exponential function.

Problems

Problem 18–1. The hyperbolic tangent function is defined by the formula $\tanh(x) = \sinh(x)/\cosh(x)$. What is the domain of $\tanh(x)$? What is the range of $\tanh(x)$? What is $\frac{d}{dx}\tanh(x)$?

Problem 18–2. Explain why the hyperbolic tangent function has an inverse function, denoted $\operatorname{arctanh}(x)$, and compute $\frac{d}{dx}\operatorname{arctanh}(x)$.

Problem 18–3. The hyperbolic secant, cosecant, and cotangent are defined by $\operatorname{sech} x = 1/\cosh x$, $\operatorname{csch} x = 1/\sinh x$, and $\coth x = 1/\tanh x$. Find the domain, range, and derivative of each of these functions.

Problem 18–4. Compute $\int_0^1 xe^x\,dx$.

Problem 18–5. Compute $\frac{d}{dw}w\ln(w)$.

Problem 18–6. Compute $\int_1^2 \ln(x)\,dx$.

Problem 18–7. True or False: For any real number x, $\arcsin(\sin(x)) = x$.

Problem 18–8. True or False: For any real number $-1 \le x \le 1$, $\cos(\arcsin(x)) = \sqrt{1-x^2}$.

Problem 18–9. A particle moves along the parabola $y = x^2$. The position $p(t)$ of the particle at time t is $p(t) = (t, t^2)$. Let $D(t)$ be the total distance travelled by the particle by time t. Use geometric reasoning to find $D'(t)$. How far does the particle travel between times $t = 0$ and $t = 4$?

Problem 18–10. The theory of radioactive decay states that the amount $A(t)$ of radioactive substance at time t decays at a rate that is proportional to the amount of substance. Denote the proportionality constant by k. Write an expression involving $A(t)$ that expresses this property. What is the sign of the proportionality constant in this equation? If $A(7) = A(0)/3$ find a formula for $A(t)$.

Problem 18–11. In recent years many politicians have vowed to "decrease the growth of government." Denote by $G(t)$ the size of government at time t. If the politicians making the previous claim were successful, what properties would you expect $G(t)$, $G'(t)$ and $G''(t)$ to have? **Briefly explain your answer.**

Solutions to Problems

Problem 18–1. Since $\cosh(x)$ is never zero, the domain of $\tanh(x)$ is all real numbers. Since $y = x$ is a slant asymptote of the unit hyperbola, the range of $\tanh(x)$ is $\{x \in \mathbf{R} : -1 < x < 1\}$. The quotient rule gives the derivative as $1/\cosh^2(x) = 1/(\cosh(x))^2 = \operatorname{sech}^2 x$.

Problem 18–2. Since the derivative of $\tanh(x)$ is always positive, $\tanh(x)$ is always increasing, and thus has an inverse function. From the general formula, $\frac{d}{dx}\operatorname{arctanh}(x) = (\cosh(\operatorname{arctanh}(x)))^2$. Dividing the fundamental relation $\cosh^2(x) - \sinh^2(x) = 1$ by $\cosh^2(x)$ gives $1 - \tanh^2(x) = 1/\cosh^2(x)$. Substituting $\operatorname{arctanh}(x)$ for x then gives $1 - x^2 = 1/\cosh^2(\operatorname{arctanh}(x))$. So finally $\frac{d}{dx}\operatorname{arctanh}(x) = 1/(1 - x^2)$. For which values of x does this formula hold?

Problem 18–3. The domain of $\operatorname{sech} x$ is $\mathbf{R}$, the range is $\{x \in \mathbf{R} : 0 < x \le 1\}$, and the derivative is $-\operatorname{sech} x \tanh x$, by the quotient rule. The domain of $\operatorname{csch} x$ is $\{x \in \mathbf{R} : x \ne 0\}$, the range is the same as the domain, and the derivative is $-\operatorname{csch} x \coth x$. The domain of $\coth x$ is $\{x \in \mathbf{R} : x \ne 0\}$, the range is $\{x \in \mathbf{R} : x < -1 \text{ or } x > 1\}$, and the derivative is $-\operatorname{csch}^2 x$.

Problem 18–4. Using integration by parts gives $\int_0^1 xe^x\,dx = e - \int_0^1 e^x\,dx = 1$.

Problem 18–5. The product rule gives the derivative as $1 + \ln(w)$.

Problem 18–6. From the previous problem, one function with derivative $\ln(x)$ is $x\ln(x) - x$. Integration by parts could also be used.

Problem 18–7. False. This is only true for $-\pi/2 \le x \le \pi/2$, the domain of the special sine function.

Problem 18–8. True. This follows from the Pythagorean identity after substituting $\arcsin(x)$ as the angle.

Problem 18–9. Here $D(t + s) = D(t) + s\sqrt{1 + (2t)^2} + o(s)$, since at time t the particle is almost traveling along a straight line with slope $2t$. Thus $D'(t) = \sqrt{1 + 4t^2}$. The Fundamental Theorem then gives $D(4) = D(4) - D(0) = \int_0^4 \sqrt{1 + 4t^2}$. The substitution rule with $f'(t) = \sqrt{1 + t^2}$ and $g(t) = 2t$ shows that this integral is $(1/2)\int_0^8 \sqrt{1 + t^2}\,dt = (1/4)\operatorname{arcsinh}(8) + 4\sqrt{1 + 8^2}$.

Problem 18–10. $A'(t) = kA(t)$. The constant k must be negative. This reflects the fact that the amount of substance is decreasing with time. Separation of variables gives $A'(t)/A(t) = k$. Integrating both sides from $t = 0$ to an arbitrary $t = w > 0$ gives $\ln(A(w)/A(0)) = kw$, from which $A(t) = A(0)e^{kt}$. The given information yields $7k = \ln(1/3)$ or $k = -\ln(3)/7$. Thus $A(t) = A(0)e^{-t\ln(3)/7}$.

Problem 18–11. I would expect the government to keep growing but at a slower rate. Hence $G'(t) \ge 0$ and $G''(t) \le 0$.

Solutions to Exercises

Exercise 18–1. An angle with radian measure t determines a sector whose area is a fraction $t/2\pi$ of the area of the unit circle. Thus the area of the sector is $(t/2\pi)(\pi(1)^2) = t/2$.

Exercise 18–2. From the graph, $\cosh(0) = 1$ and $\sinh(0) = 0$.

Exercise 18–3. The graph of $\cosh(x)$ is qualitatively similar to the graph of $1 + x^2$, while the graph of $\sinh(x)$ is qualitatively similar to the graph of x^3.

Exercise 18–4. Since $\cosh(-t) = \cosh(t)$, every point in the range of the original hyperbolic cosine function is the image of two points, one positive and one negative, in the domain of the function. One of these two points must be eliminated from the domain in order to obtain a function having an inverse function.

Exercise 18–5. Start with $\operatorname{arccosh}(\cosh(z)) = z$ and differentiate both sides to obtain $\frac{1}{\sqrt{\cosh^2(z) - 1}} \frac{d}{dz} \cosh(z) = 1$. Since $\cosh^2(z) - \sinh^2(z) = 1$, the square root term simplifies to $\sinh(z)$, and the equation re-arranges as desired.

§19. Optimization

There are many practical problems which involve selecting the best course of action: produce the most fuel efficient car, the least expensive piece of equipment, etc. When the criteria for determining the course of action can be expressed as a function of the variables under the control of the designer, the best course of action can be identified using calculus techniques.

Example 19–1. A cylindrical can, including top and bottom, is to be designed to hold 384 milliliters of liquid. How should the radius r and height h of the can be selected so that the least amount of material is used in making the can? The amount of material required to make a cylindrical can of radius r and height h is $2\pi r^2 + 2\pi rh$, which is a function of two variables. Because the volume of the can is known, the two variables are related. The volume of the cylindrical can is $\pi r^2 h$ using general properties of cylinders. The volume here is known to be 384. Hence $\pi r^2 h = 384$. This equality can be used to express either of the variables in terms of the other. Solving for h gives $h = 384/\pi r^2$. Using this, the amount $A(r)$ of material used to make a can of radius r and having volume 384 is $A(r) = 2\pi r^2 + 768/r$. This function is defined for $r > 0$ in the context of this problem. The objective is to find the value of r at which $A(r)$ takes its smallest value.

Exercise 19–1. How does the example proceed if the equation $\pi r^2 h = 384$ is solved for r in terms of h?

A good first step is to understand carefully what is meant by saying that m is a location at which a function f takes its smallest value. The point m is a location at which a function f takes is smallest value if m is in the domain of f, and $f(m) \le f(x)$ for all other points x in the domain of f. Such a point m is called a **location of the absolute minimum** of the function f, and the value $f(m)$ is called the **absolute minimum** of f. Similarly, M is a location at which f takes its largest value if M is in the domain of f and $f(M) \ge f(x)$ for all other x in the domain of f. Such a point M is called a **location of the absolute maximum** of f, and the value $f(M)$ is called the **absolute maximum** of the function f.

Exercise 19–2. Can there be more than one location at which a function takes its smallest or largest value?

Finding the location at which a function f takes its smallest or largest value can be approached geometrically. Suppose m is a location at which f takes its smallest value. Then to either the left or right of m, the values of f must be larger than the value at m. Thus to the left of m, the slope of the approximating line must be negative; to the right of m the slope of the approximating line must be positive. Hence $f'(m) = 0$. Similar reasoning can be applied if M is the location at which f

takes its largest value.

Exercise 19–3. Argue that if M is a location at which f takes its largest value then $f'(M) = 0$.

This geometric intuition is algebraically supported by the fundamental equation. Since $f(m+s) = f(m)+f'(m)s+o(s)$, for small enough s the right side of this equality is dominated by the term $f'(m)s$. If $f'(m) \neq 0$ the value of $f(m+s)$ could be made less than $f(m)$ by choosing a small enough s of sign opposite that of $f'(m)$. So $f'(m)$ must be zero. Similar reasoning applies at a location M of a maximum value of f.

Points at which the derivative is zero do not *have* to be the location of either a maximum or a minimum. Such points merely *could* be the location of a maximum or minimum. Here is an example illustrating this behavior.

Example 19–2. The function $f(x) = x^3$ on the the domain $-1 \leq x \leq 1$ has its maximum value at $x = 1$ and its minimum value at $x = -1$. Even though $f'(0) = 0$, $x = 0$ is not the location of either a maximum value or minimum value of f.

The preceding example points out that the endpoints of the domain of a function *might* be the location of a maximum or a minimum.

Example 19–3. The function $g(x) = x^{2/3}$ is defined for all real numbers x. Notice that $g(x) \geq 0$ for all x. Simple computation gives $g'(x) = (2/3)x^{-1/3}$, which is not defined at $x = 0$. But $x = 0$ is the location at which g has its minimum value.

The previous example illustrates that points at which the derivative is not defined *might* also be points at which the function takes its maximum or minimum value. Notice too that a function, such as g in the example, may not have a maximum value.

Exercise 19–4. Give an example of a function which fails to have a minimum value.

The preceding examples lead to the following method for finding the location at which a function has its maximum or minimum value. First, locate all points

(1) At which the derivative of the function is zero, or

(2) Which are endpoints of the domain of the function, or

(3) The derivative of the function is not defined.

Second, compute the value of the function at all of the identified points which are in the domain. If all of the points are in the domain, those corresponding to the largest computed value are the location(s) of the absolute maximum of the function; those corresponding to the smallest computed value are the location(s) of the absolute minimum of the function. If the function takes larger values near any of the identified points which are not in the domain, the function does not have a maximum; if the function takes smaller values near any of the identified points which are not in the domain, the function does not have a minimum.

Example 19–4. The procedure for finding the location of the absolute maximum and minimum of a function are now applied to the function $A(r) = 2\pi r^2 + 768/r$, $r > 0$, from the first example of this section. Computing gives $A'(r) = 4\pi r - 768/r^2$, which is defined for $r > 0$. The derivative is zero only when $4\pi r - 768/r^2 = 0$, that is, when $r = (768/4\pi)^{1/3}$. The endpoints of the domain of A are 0 and infinity. There are no points in the domain of A at which the derivative is not defined. For r positive and near 0, the values $A(r)$ become arbitrarily large. For r large and positive, the values of $A(r)$ also become arbitrarily large. So $(768/4\pi)^{1/3}$ is the location of the minimum of A. To minimize the material used in constructing the cylindrical can, the radius of the can should be $(768/4\pi)^{1/3}$.

Exercise 19–5. What should the height of the can made with the minimal amount of material be?

Exercise 19–6. If the problem had be solved by expressing the amount of material in terms of h instead of r, how is the solution found?

Studying the values of a function near points at which the function is not even defined is a useful idea. The concept of limit is used to make this notion more precise and usable. The **limit** of $f(x)$ as x approaches a is L, denoted $\lim_{x\to a} f(x) = L$, if L is the single value that $f(x)$ can be forced to be near by requiring x to be near, but not equal to, a. If there is no such single number, the limit does not exist.

Example 19–5. For the function $f(x) = x^2$, $\lim_{x\to 3} f(x) = 9$, since when x is near 3, x^2 is near 9. Notice that in this case the value of the limit can be computed by simply plugging in 3 for x in the formula for f.

Example 19–6. For the function $g(x) = \dfrac{x^2-1}{x-1}$, $\lim_{x\to 1} g(x) = 2$. This is because for $x \neq 1$, $\dfrac{x^2-1}{x-1} = \dfrac{(x+1)(x-1)}{x-1} = x+1$. Thus if x is near 1, $x+1$ is near 2. Notice that in this case the limit can not be computed by plugging in, since 1 is not in the domain of g.

Example 19–7. For the function $h(x) = \sqrt{x}$, $\lim_{x\to 0} h(x)$ does not exist, since for small negative x the function is not defined.

The situation of the last example is exactly the situation that occurs when studying the values of a function at the endpoints of its domain. For this reason, the concept of limit is expanded slightly. The limit of $f(x)$ as x approaches a from the right is L, denoted $\lim_{x\to a^+} f(x) = L$, if L is the single value that $f(x)$ can be forced to be near by requiring x to be near a and $x > a$. The limit of $f(x)$ as x approaches a from the left is L, denoted $\lim_{x\to a^-} f(x) = L$, if L is the single value that $f(x)$ can be forced to be near by requiring x to be near a and $x < a$.

Example 19–8. For the function $h(x) = \sqrt{x}$, $\lim_{x\to 0^+} h(x) = 0$, while $\lim_{x\to 0^-} h(x)$ does not exist.

Example 19–9. For the function $i(x) = x/|x|$, $\lim_{x\to 0^-} i(x) = -1$.

Exercise 19–7. What is $\lim_{x\to 0^+} i(x)$? What is $i(0)$?

Example 19–10. The function $\sin(1/x)$ on the domain $x > 0$ oscillates rapidly to the right of the origin. Thus $\lim_{x\to 0^+} \sin(1/x)$ does not exist, since there is no *single* value that $\sin(1/x)$ can be forced to be near for small positive values of x.

Although infinity is not a number, the concept of infinity is useful in representing values that can be made arbitrarily large.

Example 19–11. The limit $\lim_{x\to 0} \frac{1}{x^2} = \infty$, since for small positive or negative values of x the ratio $1/x^2$ is a large positive number, and can be made as large as desired by choosing x close enough to zero. Likewise $\lim_{x\to 0} \frac{-5}{x^2} = -\infty$, since in this case the ratio can be made to be an arbitrary large negative number by choosing x close enough to zero.

Exercise 19–8. Compute $\lim_{x\to 0^+} \frac{1}{x}$, $\lim_{x\to 0^-} \frac{1}{x}$, and $\lim_{x\to 0} \frac{1}{x}$.

Example 19–12. Finally, the values of a function for arbitrarily large inputs can be studied. For example, if $r(x) = \frac{x+1}{x-1}$, then $\lim_{x\to\infty} r(x) = 1$, since for large values of x the numerator and denominator are nearly equal. Similarly, $\lim_{x\to -\infty} r(x) = 1$ too.

Exercise 19–9. What is $\lim_{x\to\infty} \frac{x^2+7x}{x^2-7x}$? What is $\lim_{x\to -\infty} \frac{x^2+7x}{x^2-7x}$?

Using the concept of limit, the earlier investigation into the location of maximum and minimum values can be more succinctly completed.

Example 19–13. For the function $A(r) = 2\pi r^2 + 768/r$, the possible locations of a maximum or minimum were identified as $r = (768/4\pi)^{1/3}$, which was the point at which the derivative is zero, as well as the points 0 and ∞, which are the endpoints of the domain. Since $\lim_{r\to 0^+} A(r) = \infty$ and $\lim_{r\to\infty} A(r) = \infty$, the function $A(r)$ has no maximum value, and the minimum occurs at $r = (768/4\pi)^{1/3}$.

The concept of limit also serves to make the definition of derivative precise. The earlier discussion of $o(s)$ concluded that the precise understanding of $o(s)$ was that $\lim_{s \to 0} o(s)/s = 0$. This observation leads to an alternate definition of derivative which is often used. The derivative of the function f at the point $x = a$ can be defined by $f'(a) = \lim_{s \to 0} \dfrac{f(a+s) - f(a)}{s}$. To see that this definition is equivalent to the definition that has been used here, use the fundamental equation to write $f(a+s) - f(a) = f'(a)s + o(s)$. Substitution then gives $\lim_{s \to 0} \dfrac{f(a+s) - f(a)}{s} = \lim_{s \to 0} f'(a) + \dfrac{o(s)}{s} = f'(a)$, and the two definitions agree. The definition of derivative used here makes derivations and approximations easier to understand.

Problems

Problem 19–1. True or False: For any real number $x > 0$, $\arctan x \geq x/(1+x^2)$.

Problem 19–2. Compute $\left.\dfrac{d}{dt} e^t \sin t\right|_{t=\pi}$.

Problem 19–3. Compute $\dfrac{d}{dx} x\sqrt{1+x^2}$.

Problem 19–4. Compute $\displaystyle\int_0^{\pi} \cos 2t\, dt$.

Problem 19–5. Compute $\displaystyle\lim_{t\to 0} \frac{\sin t}{t}$.

Problem 19–6. Compute $\displaystyle\lim_{t\to \infty} \frac{\sin t}{t}$.

Problem 19–7. Compute $\displaystyle\int_0^{3} xe^{x^2}\, dx$.

Problem 19–8. Compute $\displaystyle\int_0^{3} xe^{x}\, dx$.

Problem 19–9. A box without a top is to be made using 96 square inches of cardboard. If the base of the box is square, what is the largest possible volume of the box?

Problem 19–10. A cone shaped drinking cup is to be made out of paper. The cup must hold 100 cubic centimeters of fluid. What is the height and radius of the cup holding this amount of fluid which uses the least material in its construction? If the radius of the opening of the cup is r and the height is h, the volume is $\pi r^2 h/3$ and the surface area is $\pi r\sqrt{r^2+h^2}$.

Problem 19–11. Mary's boat leaves a dock at 1:00 PM and travels due north at 10 miles per hour. Robert's boat reaches the same dock at 2:00 PM after traveling due west at 12 miles per hour. At what time were the two boats closest to each other?

Solutions to Problems

Problem 19–1. True, since $\arctan x = \int_0^x \frac{1}{1+t^2}\,dt \geq x/(1+x^2)$ by comparing the area represented by the integral with the area of a rectangle.

Problem 19–2. Using the product rule gives $\left(e^t \sin t\right)' = e^t \cos t + e^t \sin t$. Plugging in $t = \pi$ gives the value as $-e^{\pi}$.

Problem 19–3. Using the product rule and chain rule gives $\frac{d}{dx} x\sqrt{1+x^2} = x^2/\sqrt{1+x^2} + \sqrt{1+x^2} = (1+2x^2)/\sqrt{1+x^2}$.

Problem 19–4. The substitution rule gives $\int_0^{\pi} \cos 2t\,dt = (1/2)\int_0^{2\pi} \cos t\,dt = 0$.

Problem 19–5. From the definition of derivative, $\sin t = 0 + \cos(0)t + o(t)$ for t near 0. Making this substitution shows that the limit is 1.

Problem 19–6. Since $-1 \leq \sin t \leq 1$ for all t, the limit is zero.

Problem 19–7. The substitution method with $g(x) = x^2$ gives $\int_0^3 xe^{x^2}\,dx = (1/2)\int_0^9 e^x\,dx = (e^9 - 1)/2$.

Problem 19–8. Integration by parts gives the value as $\int_0^3 xe^x\,dx = 3e^3 - \int_0^1 e^x\,dx = 2e^3 + 1$.

Problem 19–9. If the side length of the base of the box is b, the volume of the box is $V(b) = hb^2$, where h is the height of the box. The amount of cardboard used to build such a box is $b^2 + 4bh$, and this must equal 96. Solving $b^2 + 4bh = 96$ for h in terms of b gives $h = 24/b + b/4$. Thus $V(b) = 24b + b^3/4$ after substitution into the earlier expression for the volume. The domain of V is $\{b \in \mathbf{R} : 0 < b < \sqrt{96}\}$. Now $V'(b) = 24 + 3b^2/4$ which is zero on this domain only when $b = \sqrt{32}$, and there are no values of b in the domain at which $V'(b)$ is not defined. Since $\lim_{b\to 0^+} V(b) = 0$ and $\lim_{b\to\sqrt{96}} V(b) = 0$, the maximum value of V is attained when $b = \sqrt{32}$. The largest volume is therefore $V(\sqrt{32}) \approx 181.019$.

Problem 19–10. Setting the volume equal to 100 gives the relationship $\pi r^2 h/3 = 100$ between r and h. Using this, the amount of paper required is $P(r) = \pi r\sqrt{r^2 + (300/\pi r^2)^2}$ in terms of r alone. The domain of P is $\{r \in \mathbf{R} : 0 < r\}$. Now $P'(r) = \pi r(2r - 360000/\pi^2 r^5)/2\sqrt{r^2 + (300/\pi r^2)^2} + \pi\sqrt{r^2 + (300/\pi r^2)^2}$ by the product rule and chain rule. Thus $P'(r) = 0$ only when $r^6\pi^2 - 45000 = 0$, or when $r = (45000)^{1/6}/\pi^{1/3} \approx 4.072$. Since $\lim_{r\to 0^+} P(r) = \infty$ and $\lim_{r\to\infty} P(r) = \infty$, this value of r corresponds to the minimal amount of paper usage. The corresponding height $h \approx 5.758$.

Problem 19–11. Using a coordinate system with origin at the dock, and measuring time t in hours past 1:00 PM gives the position of Mary's boat at time t as $(0, 10t)$ and the position of Robert's boat at time t as $(12 - 12t, 0)$. The distance between the boats is $D(t) = \sqrt{(12-12t)^2 + (10t)^2}$, for $0 \le t \le 1$. Hence $D'(t) = (488t - 288)/\sqrt{(12-12t)^2 + (10t)^2}$. The derivative is zero at time $t = 288/488$. Since $D(0) = 12$, $D(1) = 10$, and $D(288/488) \approx 7.682$, the distance is a minimum at time $t = 288/488$, that is, at approximately 1:35PM.

Solutions to Exercises

Exercise 19–1. Since $r = \sqrt{384/\pi h}$, the amount of material used as a function of h is $B(h) = 768/h + 2\sqrt{384\pi h}$, for $h > 0$. The objective is to find the value of h at which $B(h)$ is as small as possible.

Exercise 19–2. Yes. The function $\sin x$ provides an example. The smallest value -1 is obtained at $3\pi/2$, $7\pi/2, \ldots$, among others. The largest value 1 is also obtained at many places.

Exercise 19–3. Values of f both to the left and right of M must be smaller than the value of f at M. So to the left of M the slope of the approximating lines must be positive; to the right of M the slope of the approximating lines must be negative. Hence $f'(M) = 0$.

Exercise 19–4. One example is the function $h(x) = 1/(1 + x^2)$. Where does h have its maximum value?

Exercise 19–5. From the earlier discussion, the height h and radius r are related by $h = 384/\pi r^2$. So the height should be $384/\pi(768/4\pi)^{2/3}$.

Exercise 19–6. From an earlier exercise, the function whose minimum is to be found in this case is $B(h) = 768/h + 2\sqrt{384\pi h}$, for $h > 0$. Now $B'(h) = -768/h^2 + \sqrt{384\pi}/\sqrt{h}$, which is defined everywhere in the domain of B and is zero when $h = (2\sqrt{384/\pi})^{2/3}$. Since $B(h)$ takes large values for h near 0 or infinity, the minimum value of B occurs when $h = (2\sqrt{384/\pi})^{2/3}$. This is the same value as found by the other approach.

Exercise 19–7. $\lim_{x \to 0^+} i(x) = 1$, and $i(0)$ is not defined.

Exercise 19–8. $\lim_{x \to 0^+} \dfrac{1}{x} = \infty$, $\lim_{x \to 0^-} \dfrac{1}{x} = -\infty$, and $\lim_{x \to 0} \dfrac{1}{x}$ does not exist.

Exercise 19–9. $\lim_{x \to \infty} \dfrac{x^2 + 7x}{x^2 - 7x} = 1$ and $\lim_{x \to -\infty} \dfrac{x^2 + 7x}{x^2 - 7x} = 1$ too.

§20. Maclaurin and Taylor Expansions

The fundamental equation and the Fundamental Theorem of Calculus are the first steps in a more general expansion of a function in terms of its derivatives at a point.

Example 20–1. Applying the Fundamental Theorem to the function e^x gives $e^x = e^0 + \int_0^x e^t\,dt = 1 + \int_0^x e^t\,dt$. Similarly, $e^t = 1 + \int_0^t e^u\,dt$. Substituting this second expression into the first gives $e^x = 1 + \int_0^x \left(1 + \int_0^t e^u\,du\right)\,dt = 1 + x + \int_0^x \int_0^t e^u\,du\,dt$. This expansion parallels the expansion given by the fundamental equation, but is valid whether or not x is small. This substitution can also be repeated.

Exercise 20–1. What happens if the substitution is used again?

The expansion obtained in this way is called the **Maclaurin expansion** of the function e^x. The distinguishing aspect of a Maclaurin expansion is that the expansion starts at the point $x = 0$.

Exercise 20–2. Find the Maclaurin expansion of $\sin x$ up to terms involving x^3.

More generally, since the Fundamental Theorem yields $f(x) = f(a) + \int_a^x f'(t)\,dt$ and since the Fundamental Theorem also gives $f'(t) = f'(a) + \int_a^t f''(u)\,du$ substitution produces

$$\begin{aligned} f(x) &= f(a) + \int_a^x \left(f'(a) + \int_a^t f''(u)\,du\right)\,dt \\ &= f(a) + f'(a)(x-a) + \int_a^x \int_a^t f''(u)\,du\,dt. \end{aligned}$$

This procedure produces the **Taylor expansion** of the function f about the point $x = a$. The Fundamental Theorem can again be applied to f'', and this process can then be continued indefinitely.

Exercise 20–3. Find the Taylor expansion of e^x about $x = 3$.

Taylor and Maclaurin expansions can be used to refine some of the ideas developed earlier. The important points to note are that no approximation is involved in either of these expansions, and the expansions hold for all values of x, small or not, in the domain of the function being expanded.

Example 20–2. Suppose m is the location of a minimum of the function f. The Taylor expansion of f around $x = m$ is $f(x) = f(m) + f'(m)(x-m) + \int_m^x \int_m^t f''(u)\,du\,dt$, as was seen above. Now $f'(m) = 0$, and using this gives $f(x) = f(m) + \int_m^x \int_m^t f''(u)\,du\,dt$. The integral must be positive for x near m, since m is the location of a minimum of f. Conversely, if $f'(m) = 0$ and if $f''(u) \geq 0$ for *all* u then m must be the location of a minimum of f. This global property of the second derivative guarantees the existence of a minimum.

Exercise 20–4. What is the analogous statement at the location M of a maximum value of f?

A function f is **convex** if the domain of f is an interval and $f''(x) \geq 0$ at all points in the domain of f. The preceding discussion shows that the maximum value of a convex function must occur at an endpoint of the domain, and any point m at which $f'(m) = 0$ is the location of a minimum of f. A function f is **concave** if the domain of f is an interval and $f''(x) \leq 0$ at all points in the domain of f. A concave function has a maximum value at any point M at which $f'(M) = 0$ and the minimum value occurs at an endpoint of the domain.

Some calculus books refer to a convex function as 'concave up' and to a concave function as 'concave down.' This usage seems to be confined to calculus books. The terms convex and concave are more widely used.

Exercise 20–5. Show that if f is convex, the graph of f lies on or above the tangent line to the graph of f at any point.

Exercise 20–6. Show that if f is concave, the graph of f lies on or below the tangent line to the graph of f at any point.

Exercise 20–7. Show that e^{-x} is convex.

Exercise 20–8. Is $\ln x$ a concave function?

Example 20–3. As another use of Maclaurin expansions the growth rate of e^x is studied. Continuing the Maclaurin expansion of e^x in the first example of this section gives $e^x = 1 + x + x^2/2 + x^3/6 +$ a remainder integral. Now values of the exponential function are always positive, so the remainder integral must be positive when x is positive. Thus $e^x > x^3/6$ for $x > 0$, and $\lim_{x\to\infty} e^x/x^2 \geq \lim_{x\to\infty}(x^3/6)/x^2 = \infty$. So e^x increases to infinity faster than x^2 as x goes to infinity. Continuing the Maclaurin expansion provides a proof that for any positive p, $\lim_{x\to\infty} e^x/x^p = \infty$.

Exercise 20–9. What is $\lim_{x\to\infty} \dfrac{\ln x}{x}$?

As a final use of these expansion techniques the error of the earlier reconstruction method can be estimated.

Example 20–4. The fundamental equation $f(x+s) = f(x) + f'(x)s + o(s)$ was used earlier to reconstruct the graph of f from a knowledge of the derivative of f together with a single point on the graph. This reconstruction was only approximate, since the $o(s)$ term was treated as being zero. How much error did this introduce in the computation? Consider the first step in such an approximation, with the starting point being $x = 0$ and step size being 0.1. The approximation gives $f(0.1) \approx f(0) + f'(0)(0.1)$. The true value is $f(0.1) = f(0) + f'(0)(0.1) + \int_0^{0.1} \int_0^t f''(u)\,du\,dt$. The error for this one step is therefore $\int_0^{0.1} \int_0^t f''(u)\,du\,dt$. If the second derivative is trapped between two numbers, say $m \le f''(u) \le M$, direct computation gives

$$m(0.1)^2/2 \le \int_0^{0.1} \int_0^t f''(u)\,du\,dt \le M(0.1)^2/2$$

as an estimte of the size of the error for this one step. Computing a particular value of f by this method, say $f(7)$, would require about $7/(0.1)$ steps, so the total error in computing $f(7)$ would lie between $7m(0.1)/2$ and $7M(0.1)/2$. If bounds on f'' are available, the step size can be adjusted to make this error as small as desired.

Problems

Problem 20–1. True or False: For any real number x, $\cosh x = \sqrt{1 + \sinh^2 x}$.

Problem 20–2. Compute $\lim_{x\to\infty} x^2 e^{-x}$.

Problem 20–3. Find the Maclaurin expansion of $\cos z$ up to terms involving z^2.

Problem 20–4. Compute $\lim_{z\to 0} \dfrac{1 - \cos z}{z^2}$.

Problem 20–5. Compute $\dfrac{d}{dz} \sec z \Big|_{z=0}$.

Problem 20–6. Compute $\int_0^1 \cosh x \, dx$.

Problem 20–7. Compute $\int_0^2 \sqrt{1 + w^2} \, dw$.

Problem 20–8. Compute $\lim_{x\to 0^+} x \ln x$.

Problem 20–9. Let $C(v)$ denote the fuel consumption, in gallons per hour, of a car traveling with speed v in miles per hour. The function $C(v) = 1 + 0.0002(v - 45)^2$ models the fuel consumption for a particular car. At what speed should this car be driven in order to get the best gas mileage, that is, in order to use the least fuel per mile driven?

Solutions to Problems

Problem 20–1. True, since $\cosh^2 x - \sinh^2 x = 1$ and $\cosh x \geq 1$ for all x.

Problem 20–2. This limit is zero, since $\lim_{x\to\infty} e^x/x^2 = \infty$.

Problem 20–3. Here $\cos z = 1 + \int_0^z -\sin t\,dt$ and $\sin t = \int_0^t \cos u\,du$, so $\cos z = 1 + \int_0^z \left(-\int_0^t \cos u\,du\right) dt = 1 - \int_0^z \int_0^t \left(1 - \int_0^u \sin v\,dv\right) du\,dt = 1 - z^2/2 + \int_0^z \int_0^t \int_0^u \sin v\,dv\,du\,dt$.

Problem 20–4. Using the Maclaurin expansion of the previous problem gives the limit as $1/2$. A careful argument requires a bound on the remainder integral. Since $-1 \leq \sin v \leq 1$, the remainder integral satisfies the inequalities $-z^3/6 = \int_0^z \int_0^t \int_0^u (-1)\,dv\,du\,dt \leq \int_0^z \int_0^t \int_0^u \sin v\,dv\,du\,dt \leq \int_0^z \int_0^t \int_0^u 1\,dv\,du\,dt = z^3/6$. So the remainder integral is $o(z^2)$ and the ratio of the remainder integral to z^2 does vanish as $z \to 0$.

Problem 20–5. The quotient rule gives the derivative of $\sec z = 1/\cos z$ as $\sec z \tan z$. The value of the derivative at zero is therefore 0.

Problem 20–6. Since the derivative of $\sinh x$ is $\cosh x$, the Fundamental Theorem gives $\int_0^1 \cosh x\,dx = \sinh 1 - \sinh 0 = \sinh 1$.

Problem 20–7. $\int_0^2 \sqrt{1+w^2}\,dw = \sqrt{1+2^2} + (1/2)\operatorname{arcsinh} 2$ by a geometric argument used earlier.

Problem 20–8. Replace x by e^{-x} and let the new x head to infinity. This substitution gives $\lim_{x\to 0^+} x \ln x = \lim_{x\to\infty} e^{-x}(-x) = 0$.

Problem 20–9. Driving at speed v for one hour will require $C(v)$ gallons and will travel v miles. The gas mileage, $G(v)$, at this speed v is thus $v/C(v)$. The domain of $G(v)$ is $0 < v < \infty$. The quotient rule gives $G'(v) = (C(v) - vC'(v))/(C(v))^2$. Since $C(v)$ is never zero, this derivative is defined everywhere in the domain of G, and $G'(v) = 0$ only when $C(v) = vC'(v)$. Solving this quardratic equation gives $v \approx 83.81$. Since $\lim_{v\to 0^+} G(v) = 0 = \lim_{v\to\infty} G(v)$, the best gas mileage is obtained at a speed of about 84 miles per hour. Don't try this out at home!

Solutions to Exercises

Exercise 20–1. Writing $e^u = 1 + \int_0^u e^v\,dv$ and plugging in gives $e^x = 1 + x + \int_0^x \int_0^t \left(1 + \int_0^u e^v\,dv\right) du\,dt = 1 + x + x^2/2 + \int_0^x \int_0^t \int_0^u e^v\,dv\,du\,dt.$

Exercise 20–2. Since $\sin x = \int_0^x \cos t\,dt$ and $\cos t = 1 + \int_0^t -\sin u\,du$, substitution gives $\sin x = \int_0^x \left(1 + \int_0^t -\sin u\,du\right) dt = x - \int_0^x \int_0^t \sin u\,du\,dt.$ Repeating this substitution process gives $\sin x = x - x^3/6 + \int_0^x \int_0^t \int_0^u \int_0^v \sin w\,dw\,dv\,du\,dt.$

Exercise 20–3. Here $e^x = e^3 + \int_3^x e^t\,dt = e^3 + \int_3^x \left(e^3 + \int_3^t e^u\,du\right) dt = e^3 + e^3(x-3) + \int_3^x \int_3^t e^u\,du\,dt.$

Exercise 20–4. At the location of a maximum, $f(x) = f(M) + f'(M)(x-M) + \int_M^x \int_M^t f''(u)\,du\,dt = f(M) + \int_M^x \int_M^t f''(u)\,du\,dt$, since $f'(M) = 0$. So the integral must be non-positive for x near M. Moreover, if $f'(M) = 0$ and if $f''(u) \le 0$ for all u then M must be the location of a maximum of f.

Exercise 20–5. If $(a, f(a))$ is any point on the graph of f the Taylor expansion gives $f(x) = f(a) + f'(a)(x-a) + \int_a^x \int_a^t f''(u)\,du\,dt \ge f(a) + f'(a)(x-a)$. The conclusion follows since $f(a) + f'(a)(x-a)$ is the graph of the tangent line at the point $(a, f(a))$.

Exercise 20–6. If $(a, f(a))$ is any point on the graph of f the Taylor expansion gives $f(x) = f(a) + f'(a)(x-a) + \int_a^x \int_a^t f''(u)\,du\,dt \le f(a) + f'(a)(x-a)$. The conclusion follows since $f(a) + f'(a)(x-a)$ is the graph of the tangent line at the point $(a, f(a))$.

Exercise 20–7. The second derivative of e^{-x} is $e^{-x} > 0$.

Exercise 20–8. The second derivative of $\ln x$ is $-1/x^2 < 0$, so $\ln x$ is concave.

Exercise 20–9. Replace x by e^x in the limit to obtain $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{x}{e^x} = 0.$

§21. The Fundamental Theorem Examined

The Fundamental Theorem of Calculus states that $\int_a^b f'(t)\,dt = f(b) - f(a)$. The Fundamental Theorem can fail in two ways.

Example 21–1. Consider the function f which takes the value 1 over the interval $0 \le x \le 1$ and the value 2 over the interval $1 < x \le 2$. Then f' exists and is 0 except at the point 1. Taking $a = 1/2$ and $b = 3/2$ and using this fact about f' gives $\int_{1/2}^{3/2} f'(t)\,dt = 0$, while $f(3/2) - f(1/2) = 1$. The Fundamental Theorem fails here because the derivative f' failed to exist at a single point.

The other way in which the Fundamental Theorem can fail is more subtle. If the function f' does not determine an area, then the geometric argument used to derive the Fundamental Theorem will fail, and so will the theorem. To examine this issue the notion of area must be defined carefully.

Example 21–2. Functions for which f' does not determine area are not too exotic. Here is an example from the book *Counterexamples in Analysis* by Gelbaum and Olmstead. Define $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$. For $x \neq 0$ the usual differentiation rules give $f'(x) = 2x\sin(1/x^2) - (2/x)\cos(1/x^2)$. Direct computation gives $f(0+s) = 0 + s^2 \sin(1/s^2) = 0 + o(s)$, so that $f'(0) = 0$. Thus f has a derivative at every point. Inspection of the formula for $f'(x)$ shows that the derivative becomes arbitrarily large near 0. As will be seen below, this will preclude the existence of an area determined by f' over the interval $0 \le x \le 1$.

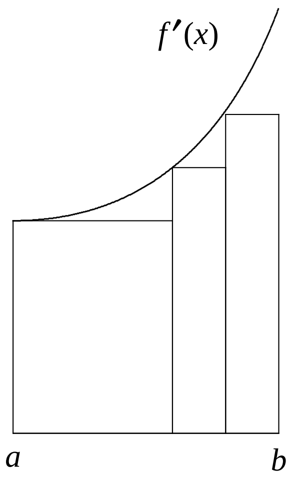
The basic geometric object for which area is defined is the square. By decomposing a rectangle into small squares, the fact that rectangles do have area and a formula for the area of a rectangle is obtained. Since two congruent right triangles can be arranged in the form of a rectangle, the fact that right triangles have area and a formula for the area of a right triangle is obtained. A general triangle can be represented as the union or difference of two right triangles, so the fact that a general triangle has area and a formula for its area can be obtained.

The simplest geometric object for which the idea of area needs clarification is the circle. Archimedes attacked the problem of finding the area of a circle by inscribing a regular n sided polygon inside the circle and incribing the circle inside a regular n sided polygon. Geometrically the area of the large and small polygon must become equal as the number n of sides tends to infinity. The common limiting value for the areas of the inscribed and inscribing polygons is the area of the circle.

In the 1850's, Bernhard Riemann used an argument similar to that of Archimedes to determine when the graph of a function determines area. His paper *Über die*

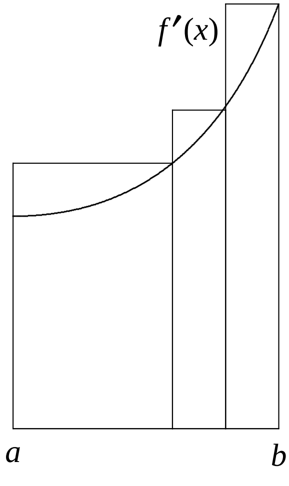
Darstellbarkeit einer Function durch eine trigonometrische Reihe appeared in 1854. An english translation is available in *Collected Works of Bernhard Riemann*, 2004. Riemann's idea was modified to a simpler, equivalent notion by Jean Gaston Darboux in 1870. Although the method below is due to Darboux, Riemann's name is often attached to the development.

The outline of Darboux's idea is this. Suppose the function f' is defined on the interval $[a,b]$. Divide the interval into a finite number of subintervals. On each subinterval, construct an inscribed rectangle which lies entirely beneath the graph of f'. The total area of all of these rectangles must be less than the area determined by f'. The situation is as in the picture below.



The total area of these rectangles is the **lower Riemann sum** corresponding to this partition of the interval of integration.

For the same subdivision of the interval, construct inscribing rectangles which enclose the graph of f'. The area of these rectangles must be at least as much as the area determined by f'. The situation is as shown below.



The total area of these rectangles is the **upper Riemann sum** corresponding to this partition of the interval of integration.

Now let the number of subdivisions increase in such a way that the length of the largest single subdivision approaches zero. If the lower and upper Riemann sums become equal in this limit, then the common limiting value is the area enclosed by the graph of f'. If lower and upper Riemann sums do not become equal in the limit, then the graph of f' does not determine an area.

One consequence of this approach is that an *unbounded function*, that is, a function whose range is not contained in a finite interval, can not determine area. This is so for the simple reason that the inscribing rectangles can not all be drawn, since the top of at least one of these is at infinity.

Exercise 21–1. Explain why the function f' of the earlier example does not determine area over the interval $0 \le x \le 1$.

The oscillatory nature of the derivative in the earlier example illustrates another, more exotic, way in which a function can fail to determine area. If the function values oscillate wildly in the vicinity of enough points, the inscribed and inscribing rectangles will never have nearly equal area. This argument shows that even bounded functions can fail to determine area.

On a more positive note, geometric reasoning leads to a simple criterion which guarantees that the graph of a function determines area. Examine a single interval in the partition. The area of the inscribed and inscribing rectangle will be nearly equal if the maximum value and minimum value of the function on this single partitioning interval are nearly equal. This will be guaranteed to occur if at every point x in the interval of integration, $\lim_{z \to x} f'(z) = f'(x)$.

This property of a function is given a name. A function g is **continuous** at the point x if x is in the domain of g and $\lim_{z \to x} g(z) = x$. The function g is continuous on a given interval if g is continuous at each point in the given interval.

The upshot of this discussion is that if the integrand is continuous on the interval of integration, then the graph of the integrand determines area. The Fundamental Theorem will then hold.

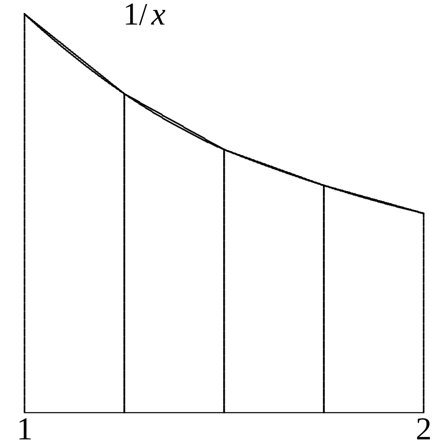
Example 21–3. Even though the function f given in the first example of this section is not continuous at $x = 1$, the graph of f does determine area. This is because the graph of f determines exactly two rectangles.

Exercise 21–2. Explain why this function f is not continuous at $x = 1$.

The geometric ideas embodied in the arguments of Darboux and Riemann lead to a different method of approximating the value of an integral.

Example 21–4. How can $\ln(2) = \int_1^2 1/x\,dx$ be computed? Since the logarithm function is defined by an integral, computing this value by using the Fundamental Theorem is impossible. A numerical method based on approximating the area represented by the integral with the area of trapezoids can be easily developed.

The idea behind the **trapezoid rule** approximation to the value of the integral is illustrated below. Subdivide the interval $1 \le x \le 2$ into 4 equal length subintervals. On each subinterval, erect a trapezoid having two vertices lying on the graph of $1/x$. The total area of these 4 trapezoids should be a good approximation to the value of the integral.



The picture indicates that the trapezoidal approximation should be quite accurate.

The detailed calculation for the approximation is

$$\begin{aligned}\int_1^2 1/x\,dx \approx{} & \frac{1}{4}(1/1 + 1/1.25)/2 \\ & + \frac{1}{4}(1/1.25 + 1/1.5)/2 \\ & + \frac{1}{4}(1/1.5 + 1/1.75)/2 \\ & + \frac{1}{4}(1/1.75 + 1/2)/2 \\ \approx{} & 0.6970\end{aligned}$$

using the formula for the area of a trapezoid.

Intuitively, increasing the number of subintervals should increase the accuracy of the approximation. In order to conveniently discuss a general number of subdivisions, some notation is required in order to present the resulting computational formula.

Formulas involving sums of terms arise in this, and many other situations. Often the terms to be added exhibit a simple pattern. In such cases a shorthand notation can be used for the sum. If g is a function whose domain is an interval of integers, say $m \leq i \leq n$, the notation $\sum_{i=m}^{n} g(i) = g(m) + g(m+1) + \ldots + g(n)$. Notice that the **index of summation** i has the **lower endpoint of summation** m as its first value. The first term to be added is the value of the function $g(m)$. The index of summation is then increased by 1, and the second term to be added is $g(m + 1)$. The index is sucessively increased by 1 until i is assigned the value specified by the **upper limit of summation** n. The last value added is $g(n)$.

Example 21–5. The sum $\sum_{i=1}^{5} i = 1+2+3+4+5 = 15$. Here the index of summation i initially takes the value 1 and is increased by 1 until the value $i = 5$.

Example 21–6. The sum $\sum_{j=-3}^{2} j^2 = (-3)^2 + (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 = 19$.

Exercise 21–3. What is $\sum_{k=2}^{5} \sin(k\pi/2)$?

Exercise 21–4. Compute $\sum_{j=1}^{3} j^3$.

Example 21–7. Using this notation, the trapezoid rule approximation to $\int_1^2 1/x\,dx$ using n subdivisions of the interval $1 \leq x \leq 2$ is $\frac{1}{n}\sum_{i=0}^{n-1} \frac{1/(1+i/n) + 1/(1+(i+1)/n)}{2}$.

Exercise 21–5. Write the trapezoid rule approximation to $\int_0^1 \sin(x^2)\,dx$ using 10 subintervals in summation notation.

Intuitively, the error in the trapezoid approximation should depend on the size of the second derivative of the integrand. An interesting, but somewhat complicated, integration by parts argument can be used to show that this is indeed the case. The details may be found elsewhere.

Problems

Problem 21–1. Compute $\sum_{k=-2}^{3}(2k-7)$.

Problem 21–2. Find the Maclaurin expansion for $\tan z$ up to terms involving z^2.

Problem 21–3. Compute $\lim_{w\to 0} \frac{\tan w}{w}$.

Problem 21–4. Compute $\frac{d}{dr} e^r \cos r\Big|_{r=\pi}$.

Problem 21–5. Compute $\int_1^3 \frac{z}{1+z^2}\,dz$.

Problem 21–6. Compute $\int_0^{1/4} \sqrt{1-t^2}\,dt$.

Problem 21–7. Water flows into a spherically shaped tank at a constant rate of 20 liters per minute. The tank is empty at time $t = 0$. Denote by $D(t)$ the depth of water in the tank at time t. Is $D'(t)$ positive, negative, or zero? Denote by t_1, t_2, and t_3 the times at which the tank is 1/3 full, 1/2 full, and 2/3 full respectively. Which of $D''(t_1)$, $D''(t_2)$ and $D''(t_3)$ are positive? Which are negative? Which are zero?

Problem 21–8. Find the maximum and minimum value of $g(w) = we^{-w^2}$ on the interval $0 < w < \infty$.

Solutions to Problems

Problem 21–1. $\sum_{k=-2}^{3}(2k-7) = -11-9-7-5-3-1 = -36.$

Problem 21–2. The derivative of $\tan z$ is $\sec^2 z$, and the derivative of $\sec^2 z$ is $2\sec^2 z \tan z$. This gives the Maclauring expansion as $\tan z = 0 + z + 0z^2 +$ an integral remainder.

Problem 21–3. Using the Maclaurin expansion of the preceding problem shows that the limit is 1.

Problem 21–4. Here $\frac{d}{dr}e^r \cos r = -e^r \sin r + e^r \cos r$, so the value when $r = \pi$ is $-e^{\pi}$.

Problem 21–5. Using substitution wih $g(z) = 1 + z^2$ gives the value as $(1/2)\int_2^{10} 1/\sqrt{z}\,dz = \sqrt{10} - \sqrt{2} \approx 1.7480.$

Problem 21–6. $\int_0^{1/4} \sqrt{1-t^2}\,dt = (1/2)(1/4)\sqrt{1-(1/4)^2} + (1/2)\arcsin(1/4)$ by an earlier geometric argument.

Problem 21–7. Here $D'(t)$ is positive until the time at which the tank is full, because the amount of water in the tank is increasing. When the tank is 1/3 full the surface of the tank gets larger as the tank fills. Consequently the depth of water, while increasing, increases at a slower rate as the tank fills. So $D''(t_1) < 0$. Similar reasoning shows that $D''(t_2) = 0$ and $D''(t_3) > 0$.

Problem 21–8. Here $g'(w) = (1-2w^2)e^{-w^2}$, so $g'(w) = 0$ on the domain given only when $w = \sqrt{1/2}$. The endpoints of the domain are 0 and infinity, and neither of these points lie in the domain. There are no points at which g' is undefined. Computing values gives $g(\sqrt{1/2}) = e^{-1/2}/\sqrt{2}$, $\lim_{w\to 0^+} = 0$ and $\lim_{w\to\infty} g(w) = 0$. So the maximum value of g is $e^{-1/2}/\sqrt{2}$ and g has no minimum value on this domain.

Solutions to Exercises

Exercise 21–1. For $x \neq 0$ $f'(x) = 2x\sin(1/x^2) - (2/x)\cos(1/x^2)$. For x near zero, the second term becomes arbitrarily large in both the positive and negative directions. No inscribing rectangle can possibly be drawn over the subinterval $0 \le x \le c$ for any $c > 0$. So the rectangle scheme can not be carried out.

Exercise 21–2. Since f takes the value 1 to the left of $x = 1$ and the value 2 to the right of $x = 1$, $\lim_{z \to 1} f(z)$ does not exist. So in particular the limit does not equal the value $f(1) = 1$.

Exercise 21–3. $\sum_{k=2}^{5} \sin(k\pi/2) = \sin(2\pi/2) + \sin(3\pi/2) + \sin(4\pi/2) + \sin(5\pi/2) = 0 - 1 + 0 + 1 = 0$.

Exercise 21–4. $\sum_{j=1}^{3} j^2 = 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$.

Exercise 21–5. The sum $(1/10) \sum_{j=0}^{9} \dfrac{\sin((i/10)^2) + \sin((i+1)/10)^2)}{2}$ is the trapezoid approximation to $\int_0^1 \sin(x^2)\,dx$.

§22. Some Practical Considerations

The problem of optimization involves finding the points at which the derivative of a function is zero. In simple examples these points can be found by applying the usual algebraic techniques. In more realistic problems, solving such an equation can present real difficulties.

Example 22–1. A particle is moving on an elliptical orbit in the plane. The position of the particle at time t is $p(t) = (4\cos t, 5\sin t)$. At what times is the particle closest to the point $(1, 2)$? The distance from the particle to the point at time t is $D(t) = \sqrt{(4\cos t - 1)^2 + (5\sin t - 2)^2}$. Since the sine and cosine functions are periodic with period 2π, the function $D(t)$ is also periodic with period 2π. So there will be infinitely many times t at which the particle is closest to the given point. To refine the question, at what time t, $0 \le t \le 2\pi$, is the particle closest to the given point? Computing and simplifying gives $D'(t) = \dfrac{9\sin t\cos t + 4\sin t - 10\cos t}{\sqrt{(4\cos t - 1)^2 + (5\sin t - 2)^2}}$. The derivative $D'(t)$ is defined on the entire interval, and is zero only when the numerator of the expression for $D'(t)$ is zero. From the physical properties of the orbit, the time at which the particle is closest lies in the interval $0 \le t \le \pi/2$. At which value of t in this interval is $9\sin t\cos t + 4\sin t - 10\cos t = 0$?

Simple algebraic techniques fail to provide a solution of the foregoing equation. Two numerical methods will now be described that lead to an *approximate* solution of the equation.

The first approach to solving the equation is called the **bisection method**. This approach hinges on the observation that if the value of a continuous function is less than 0 at one point and greater than zero at another, there must be at least one intermediate point at which the function is zero.

Example 22–2. The bisection method is applied to find a value of t in the interval $0 \le t \le \pi/2$ at which $N(t) = 9\sin t\cos t + 4\sin t - 10\cos t$ is zero. In this case $N(0) = -10$ and $N(\pi/2) = 4$, so there must be at least one value of t in this interval at which $N(t) = 0$. In fact, there is exactly one such value of t in this case, due to physical considerations. Now compute the midpoint of this interval. The midpoint is $t = \pi/4$. The value $N(\pi/4) = 9/2 - 3\sqrt{2} = 0.2573 > 0$. Since $N(0) = -10$, the value of t for which $N(t) = 0$ actually lies in the interval $0 \le t \le \pi/4$. Now compute the midpoint of this new interval. The midpoint is $\pi/8$. The value $N(\pi/8) = -4.526$. Since $N(\pi/4) > 0$, the value of t at which N takes the value 0 must lie in the interval $\pi/8 \le t \le \pi/4$. Continuing in this way, the value of t at which $N(t) = 0$ can be found to arbitrary accuracy.

Exercise 22–1. What is the midpoint of this last interval, and what are the endpoints of the new interval in which the desired value of t must lie?

The computations are summarized in the following table.

Endpoints						
Left(L)	Right(R)	Midpoint(M)	$N(L)$	$N(R)$	$N(M)$	$R-L$
0	1.5707	0.7853	-10	4	0.2573	1.5707
0	0.7853	0.3926	-10	0.2573	-4.526	0.7853
0.3926	0.7853	0.5890	-4.5260	0.2573	-1.934	0.3926
0.5890	0.7853	0.6872	-1.9349	0.2573	-0.778	0.1963
0.6872	0.7853	0.7363	-0.7789	0.2573	-0.244	0.0981
0.7363	0.7853	0.7608	-0.2449	0.2573	0.0102	0.0490
0.7363	0.7608	0.7485	-0.2449	0.0102	-0.116	0.0245
0.7485	0.7608	0.7547	-0.1163	0.0102	-0.052	0.0122
0.7547	0.7608	0.7577	-0.0527	0.0102	-0.021	0.0061
0.7577	0.7608	0.7593	-0.0211	0.0102	-0.005	0.0030
0.7593	0.7608	0.7600	-0.0054	0.0102	0.0024	0.0015
0.7593	0.7600	0.7597	-0.0054	0.0024	-0.001	0.0007

The value of t at which $N(t) = 0$ is $t = 0.759$, to 3 decimal places.

The bisection method is easy to apply and depends only on finding endpoints of the initial interval at which the function under consideration takes values of opposite sign. In most problems these endpoints can be determined by physical characteristics of the problem at hand, as was the case in the example. Notice that if there is more than one value which makes the function zero in that initial interval, the bisection method will find only one of them.

Exercise 22–2. What would happen if the bisection method were used to find the solution of $x^2 = 0$?

A more sophisticated approach is **Newton's Method**. To find the value of t so that $N(t) = 0$, make an initial estimate of t, say r. If this intial estimate is not a solution, take as a new, refined, estimate the intercept of the tangent line to N through the point $(r, N(r))$. Since the tangent line through $(r, N(r))$ has slope $N'(r)$, the equation of the tangent line is $y - N(r) = N'(r)(x - r)$. Thus the intercept, and new estimate, is $r - N(r)/N'(r)$. Repeat this procedure until the desired accuracy is obtained.

Example 22–3. In the context of the preceding example, $N(t) = 9\sin t\cos t + 4\sin t - 10\cos t$ and $N'(t) = 9\cos^2 t - 9\sin^2 t + 4\cos t + 10\sin t$. A reasonable first estimate is $r = 0$. Since $N(0) = -10$ and $N'(0) = 13$, the second estimate is $r = 0.7692$. The computations are summarized in the following table.

r	$N(r)$	$N'(r)$
0.0000	-10.0000	13.0000
0.7692	0.0955	10.1206
0.7598	-0.0006	10.2483
0.7599	0.0000	10.2475
0.7599	0.0000	10.2475

Thus the first time of closest approach of the particle to the point is 0.7599 to 4 decimal places.

While Newton's method often requires less computation to find the solution to the desired level of accuracy, the method can also fail to find any root, or may find a root outside the original interval of interest. These problems arise whenever the tangent line at the initial or any subsequent estimate does not approximate the function closely enough. The bisection method is a better general method.

Problems

Problem 22–1. Use the bisection method to find the value of t at which the particle is farthest from the point $(1, 2)$ to 3 decimal place accuracy.

Problem 22–2. Use Newton's method to find the value of t at which the particle is farthest from the point $(1, 2)$ to 3 decimal place accuracy.

Problem 22–3. If you haven't already done so, use an initial estimate of π in the preceding problem. Why does Newton's method produce the time of nearest approach in this case?

Problem 22–4. A cylindrical can of radius r and height h, including top and bottom, is to be constructed. The can must hold 384 milliliters of fluid. The material used to construct the can costs \$0.001 per square centimeter. Joining the top to the can requires special processing that costs \$0.002 per linear centimeter of the circumference of the top. What are the dimensions of the can with the least cost?

Problem 22–5. Compute $\int_{-5}^{7} t(t^2 + 1)\, dt$.

Problem 22–6. Compute $\int_{0}^{1} z \cosh z^2\, dz$.

Problem 22–7. True or False: $\int_{0}^{1} e^{z^2}\, dz = e - \int_{1}^{e} \sqrt{\ln(y)}\, dy$.

Problem 22–8. Compute $\left.\dfrac{d}{dy} e^y \cos y\right|_{y=0}$.

Problem 22–9. Compute $\lim_{t\to\infty} t \sin(1/t^2)$.

Problem 22–10. Compute $\lim_{x\to 0^+} x^{1/x}$.

Problem 22–11. Compute $\dfrac{d}{dt} \dfrac{t + \sin t}{t + \cos t}$. For what values of t is your formula valid?

Problem 22–12. Suppose $r(t)$ is the rate, in barrels per day, of oil consumption in the United States at time t in years, with $t = 0$ corresponding to January 1, 2002. What does $\int_{1}^{3} r(t)\, dt$ represent?

Solutions to Problems

Problem 22–1. The initial endpoints were taken as $\pi/2$ and 2π. Other choices are possible. The computations are summarized in the following table.

Endpoints						
Left (L)	Right (R)	Midpoint (M)	$N(L)$	$N(R)$	$N(M)$	$R-L$
1.5708	6.2832	3.9270	4.0000	-10.0000	8.7426	4.7124
3.9270	6.2832	5.1051	8.7426	-10.0000	-10.7043	2.3562
1.5708	5.1051	3.3379	4.0000	-10.7043	10.7496	3.5343
3.3379	5.1051	4.2215	10.7496	-10.7043	4.9279	1.7671
4.2215	5.1051	4.6633	4.9279	-10.7043	-3.0634	0.8836
4.2215	4.6633	4.4424	4.9279	-3.0634	1.1255	0.4418
4.4424	4.6633	4.5529	1.1255	-3.0634	-0.9491	0.2209
4.4424	4.5529	4.4976	1.1255	-0.9491	0.0969	0.1104
4.4976	4.5529	4.5252	0.0969	-0.9491	-0.4243	0.0552
4.4976	4.5252	4.5114	0.0969	-0.4243	-0.1632	0.0276
4.4976	4.5114	4.5045	0.0969	-0.1632	-0.0330	0.0138
4.4976	4.5045	4.5011	0.0969	-0.0330	0.0320	0.0069
4.5011	4.5045	4.5028	0.0320	-0.0330	-0.0005	0.0035
4.5011	4.5028	4.5019	0.0320	-0.0005	0.0157	0.0017
4.5019	4.5028	4.5024	0.0157	-0.0005	0.0076	0.0009
4.5024	4.5028	4.5026	0.0076	-0.0005	0.0036	0.0004

The first time t at which the particle is farthest from $(1,2)$ is $t = 4.502$, to 3 decimal places.

Problem 22–2. Here a reasonable initial estimate is $3\pi/2$. The computations then proceed as follows.

r	$N(r)$	$N'(r)$
4.7124	-4.0000	-19.0000
4.5019	0.0173	-18.8290
4.5028	0.0000	-18.8341
4.5028	0.0000	-18.8341

Thus the first time the particle is farthest from $(1,2)$ is $t = 4.502$, to 3 decimal places. What happens if the initial estimate is π?

Problem 22–3. A graph of the function $N(t)$ shows that the tangent line to the curve when $t = \pi$ has an intercept near the time of closest approach. The method then leads to the solution found earlier.

Problem 22–4. The volume requirement implies that $\pi r^2 h = 384$. The cost, in terms of r and h, of constructing a can is $(0.001)(2\pi r^2 + \pi r^2 h) + (0.002)(2\pi r)$. Using the volume requirement to eliminate h gives the cost $C(r)$ in terms of r alone as $C(r) = (0.001)(2\pi r^2 + 768/r) + (0.002)(2\pi r)$. The domain of C is $0 < r < \infty$. Computing gives $C'(r) = (0.001)(4\pi r - 768/r^2) + (0.002)(2\pi)$. The derivative is zero only when r satisfies $r^3 + r^2 - 768/4\pi = 0$. Solving using the bisection method or Newton's method gives $r = 3.632$. The derivative is defined everywhere in the domain of C. Since $\lim_{r\to 0^+} C(r) = \infty$ and $\lim_{r\to\infty} C(r) = \infty$, $r = 3.632$ corresponds to the minimum value of C. The height of the cheapest can is $h = 384/(\pi(3.632)^2)$.

Problem 22–5. One approach is to multiply out the integrand and use the power rule. The substitution rule can also be used. Either way, the value is 456.

Problem 22–6. $\int_0^1 z \cosh z^2\, dz = (1/2) \int_0^1 \cosh z\, dz = (1/2)\sinh(1)$, by using the substitution method.

Problem 22–7. Drawing a picture shows that the total area given by the two integrals adds up to the area of a rectangle, making the equality correct.

Problem 22–8. $\left.\frac{d}{dy} e^y \cos y\right|_{y=0} = (-e^y \sin y + e^y \cos y)\big|_{y=0} = 1.$

Problem 22–9. For large t, $1/t^2$ is small. Thus $\sin(1/t^2) = 0 + 1/t^2 + o(1/t^2)$ by the basic equality. Making this substitution shows that $t \sin(1/t^2) = 1/t + o(1/t)$, making the limit zero.

Problem 22–10. Recall that $x^{1/x} = e^{(1/x)\ln(x)}$, so the problem is really to find $\lim_{x \to 0^+} (1/x)\ln(x)$. Replacing x by e^{-x} and letting the new $x \to \infty$ shows that $\lim_{x \to 0^+} (1/x)\ln(x) = \lim_{x \to \infty} -x/e^{-x} = -\infty$. So the original limit is zero.

Problem 22–11. The derivative is $\frac{(t + \cos t)(1 + \cos t) - (t + \sin t)(1 - \cos t)}{(t + \cos t)^2}$, by the quotient rule. This formula is valid as long as $t + \cos t \neq 0$.

Problem 22–12. The integral is the total amount of oil, in barrels, used during the years 2003 and 2004.

Solutions to Exercises

Exercise 22–1. The new midpoint is $3\pi/16$. Since $N(3\pi/16) = -1.934$, the value of t being sought must lie in the interval $3\pi/16 \le t \le \pi/4$.

Exercise 22–2. The method would be unable to start, since all values of the function x^2 are non-negative.