The Spectrum of an Element in a Complex Banach Algebra

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1 Introduction

A self contained proof is given here of the fact that any element a of a complex Banach algebra has a non-empty spectrum. A proof given in *The Spectrum in a Banach Algebra* by Dinesh Singh, which appeared in the *American Mathematical Monthly* of October 2006, relied on integration theory for Banach valued functions. Here only a few basic facts are needed.

A complex Banach algebra is a normed vector space over the complex numbers which is also closed under the operation of multiplication of vectors and for which the norm satisfies $\|ab\| \le \|a\|\,\|b\|$ for all vectors a and b. The spectrum of an element a is the set of complex numbers z for which $a - zI$ is not invertible. Here I is the identity element of the algebra.

In outline, the argument is this. If the spectrum of the element a is empty the integral $\int_0^{2\pi} (a - re^{i\theta})^{-1}\,d\theta$ should have a value not depending on r, by analogy with complex function theory. But when $r = 0$ the value is a^{-1} while for large r the value is near zero. The contradiction that $a^{-1} = 0$ shows that the assumption of empty spectrum is false. To avoid using calculus for Banach valued functions, the integral will be replaced by a sum, but the gist of the argument will be the same.

A fact that will be used multiple times below is that if x is an element of the Banach algebra and $\|x\| < 1$ then $(I - x)^{-1}$ exists and $(I - x)^{-1} = \sum_{k=0}^{\infty} x^k$. The proof of this fact is as follows. Since $\| x \| < 1$, the partial sums of the series form a Cauchy sequence. Since a Banach space is complete, this Cauchy sequence converges. Thus the infinite series does indeed define an element of the space. Now for any n, $\| (I-x) \sum_{k=0}^{\infty} x^k - I \| \leq \| (I-x) \sum_{k=0}^{n} x^k - I \| + \| (I-x) \sum_{k=n+1}^{\infty} x^k \| \leq \| x \|^{n+1} + \| I-x \| \| x \|^{n+1} / (1 - \| x \|)$, which can be made arbitrarily small.

2 The Proof

If $a = 0$, then $z = 0$ is in the spectrum. Suppose now that $a \neq 0$ and the spectrum of a is empty. This implies that $(a-zI)^{-1}$ exists for any complex number z.

The first step is to show that $(a-zI)^{-1}$ is a continuous function of z. For any complex w with $\| (z-w)(a-zI)^{-1} \| < 1$,

$$\begin{aligned}
(a-zI)^{-1} - (a-wI)^{-1} &= (a-zI)^{-1} - (a-zI+zI-wI)^{-1} \\
&= (a-zI)^{-1}(I - (I - (z-w)(a-zI)^{-1})) \\
&= (a-zI)^{-1} \left(I - \sum_{k=0}^{\infty} \left((z-w)(a-zI)^{-1}\right)^k \right) \\
&= -(a-zI)^{-1} \sum_{k=1}^{\infty} \left((z-w)(a-zI)^{-1}\right)^k \\
&= (w-z)(a-zI)^{-2} \sum_{k=0}^{\infty} \left((z-w)(a-zI)^{-1}\right)^k .
\end{aligned}$$

Thus $\| (a-zI)^{-1} - (a-wI)^{-1} \| \leq |w-z| \| (a-zI)^{-1} \|^2 / (1 - |z-w| \| (a-zI)^{-1} \|)$ which goes to zero as $w \to z$, and continuity is proved.

The second step is to show that $\| (a-zI)^{-1} \| \to 0$ as $|z| \to \infty$. If $|z| > 2\| a \|$ then $(a-zI)^{-1} = -z^{-1}(I - az^{-1})^{-1} = -z^{-1} \sum_{k=0}^{\infty} (az^{-1})^k$. Thus $\| (a-zI)^{-1} \| \leq 1/|z|$ when $|z| > 2\| a \|$, establishing the convergence. This explicit bound will also be used in the next step and again later.

The third step is to show that $\| (a-zI)^{-1} \|$ is a bounded function of z. By continuity, the function $\| (a-zI)^{-1} \|$ is bounded by a constant B' on the compact set $|z| \leq 2\| a \|$. Hence if $B = B' + 1/2\| a \|$, $\| (a-zI)^{-1} \| \leq B$ for all complex z by virtue of the bound in the second step.

The fourth step uses simple algebra to deduce that $(a-zI)^{-1}-(a-wI)^{-1} = (z-w)(a-zI)^{-1}(a-wI)^{-1}$. Iterating this fact gives

$$\begin{aligned}
(a-zI)^{-1}-(a-wI)^{-1} &= (z-w)(a-zI)^{-1}(a-wI)^{-1} \\
&= (z-w)(a-zI)^{-1}((a-zI)^{-1}+(a-wI)^{-1}-(a-zI)^{-1}) \\
&= (z-w)(a-zI)^{-2}+(z-w)(a-zI)^{-1}(w-z)(a-zI)^{-1}(a-wI)^{-1} \\
&= (z-w)(a-zI)^{-2}-(z-w)^2(a-zI)^{-2}(a-wI)^{-1} \\
&= (z-w)(a-zI)^{-2}+E(z,w)
\end{aligned}$$

where $E(z,w) = -(z-w)^2(a-zI)^{-2}(a-wI)^{-1}$. This equality will be used reading from left to right and also from right to left. Notice that by the third step, $\| E(z,w) \| \leq B^3|z-w|^2$.

Now for the main part of the proof. For any $r \geq 0$ and any positive integer n set

$$F(r) = \frac{1}{n}\sum_{k=1}^{n}\left(a-re^{2\pi ik/n}I\right)^{-1}.$$

For any fixed n, $F(r) \to 0$ as $r \to \infty$ because of the second step. The objective is to estimate the difference between $F(r)$ and $F(0) = a^{-1}$ and show that this difference can be made arbitrarily small. The resulting contradiction will complete the proof.

For any positive integer m, telescoping, the fourth step from left to right, the fourth step from right to left, and telescoping give

$$\begin{aligned}
F(r)-F(0) &= \sum_{j=1}^{m} F\left(r\frac{j}{m}\right)-F\left(r\frac{j-1}{m}\right) \\
&= \frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\left(a-r\frac{j}{m}e^{2\pi ik/n}I\right)^{-1}-\left(a-r\frac{j-1}{m}e^{2\pi ik/n}I\right)^{-1} \\
&= \frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{re^{2\pi ik/n}}{m}\left(a-r\frac{j}{m}e^{2\pi ik/n}I\right)^{-2}+E\left(r\frac{j}{m}e^{2\pi ik/n},r\frac{j-1}{m}e^{2\pi ik/n}\right) \\
&= \frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{\left(a-r\frac{j}{m}e^{2\pi ik/n}I\right)^{-1}-\left(a-r\frac{j}{m}e^{2\pi i(k-1)/n}I\right)^{-1}-E\left(r\frac{j}{m}e^{2\pi ik/n},r\frac{j}{m}e^{2\pi i(k-1)/n}\right)}{1-e^{-2\pi i/n}} \\
&\quad +\frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n}E\left(r\frac{j}{m}e^{2\pi ik/n},r\frac{j-1}{m}e^{2\pi ik/n}\right) \\
&= \frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\frac{-E\left(r\frac{j}{m}e^{2\pi ik/n},r\frac{j}{m}e^{2\pi i(k-1)/n}\right)}{1-e^{-2\pi i/n}}+\frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n}E\left(r\frac{j}{m}e^{2\pi ik/n},r\frac{j-1}{m}e^{2\pi ik/n}\right)
\end{aligned}$$

Using now the bound on the E function from the fourth step,

$$\begin{aligned}\|F(r)-F(0)\| &\leq \frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n} B^3 r^2 |1-e^{-2\pi i/n}| j^2/m^2 + \frac{1}{n}\sum_{j=1}^{m}\sum_{k=1}^{n} B^3 r^2/m^2 \\ &\leq B^3 r^2 |1-e^{-2\pi i/n}| \sum_{j=1}^{m} j^2/m^2 + B^3 r^2/m.\end{aligned}$$

Making use of this estimate yields, for $r \geq 2\|a\|$,

$$\begin{aligned}\|a^{-1}\| &= \|F(0)\| \\ &\leq \|F(0)-F(r)\| + \|F(r)\| \\ &\leq B^3 r^2 \frac{1}{n}\sum_{j=1}^{m} j^2/m^2 + B^3 r^2/m + 1/r.\end{aligned}$$

Letting $n \to \infty$, then $m \to \infty$, and then $r \to \infty$ shows that $\|a^{-1}\| = 0$, and thus that $a^{-1} = 0$, which is impossible. The proof is complete.