Lecture Notes on Algebra and Trigonometry

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§0. Preface

The objective of a large part of mathematics is to study the relationships that exist between variables. There are essentially two approaches to doing this: visually and through formulas. Both of these methods will be explored in these notes. While the term 'formula' suggests a list of equations to memorize, this is not the case. There are only a few important ideas in the study of algebra and trigonometry which are used repeatedly and in different contexts. Understanding these essential ideas, and being able to apply them, is the goal of these notes.

A secondary objective is to be able to translate a problem stated in words into a mathematical problem, and translate the solution of the corresponding mathematical problem back into words. Mathematics exists substantially to solve problems that arise in the real world. It is only through the ability to perform this translation that real use can be made of the power of mathematics.

Throughout these notes are various exercises and problems. The reader should attempt to work all of these. Solutions, sometimes in the form of hints, are provided for most of the exercises and problems.

These notes also constitute an attempt to identify the essential elements of algebra and trigonometry and to separate these elements from purely computational and formal manipulations which can now be done by computers. The emphasis here is therefore on conceptual understanding of certain key elements, and not on encyclopedic knowledge.

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§1. Sets

If the objective of mathematics is to find and study the relationship between variables, then a reasonable first step is to describe where the values of these variables must be. To this end, the idea of a set is reviewed and some commonly used sets are introduced.

A **set** is simply a collection of objects. One might speak of the set of students in this classroom, the set of bicycles on campus, and so on. An individual object in a set is called an **element** of the set. In mathematics, sets often consist of numbers.

Example 1–1. The set of **natural numbers**, which is denoted by **N**, is the collection whose elements are 1, 2, 3, ... The **integers**, denoted by **Z**, is the set whose elements are ...,–3, –2, –1, 0, 1, 2, 3, ... The **rational numbers**, denoted by **Q**, is the set of all numbers which can be written as the ratio of two integers.

Exercise 1–1. Is 5.96 a rational number? Is $\sqrt{2}$ a rational number?

Exercise 1–2. Is 3.333... a rational number?

Certainly any decimal that terminates is a rational number. It is not difficult to show that any repeating decimal number is a rational number. It is also not difficult to give an example of a decimal number which does not repeat and does not terminate. It can be shown, but will not be shown here, that a non-repeating, non-terminating, decimal number is not rational. The collection of all numbers which can be written in decimal form (repeating or not) is the set of **real numbers**, and is denoted by **R**.

Example 1–2. The number 3.12345678910111213... is a real number which is not a rational number.

Exercise 1–3. Is every rational number a real number?

Giving sets a visual representation is often very useful. The visual representation of the set of real numbers **R** is as a straight, infinite, line. The individual numbers (elements) are located along this line.

Typically one of these sets above is too large a location for the possible values of a variable. Additional information narrows the set of possible values to a piece, that is a **subset**, of the original set. The subset is specified notationally by giving the condition required to be an element of the subset.

Example 1–3. The set of real numbers which are at least 3 is written notationally as $\{x \in \mathbf{R} : x \geq 3\}$. The notation is read as "the set of x in the real numbers such that

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x is greater than or equal to 3." In this notation the colon is read as "such that" or "with the property that." The notation $x \in \mathbf{R}$ means that the number x is an element of the set of real numbers. The inequality following the colon gives the additional property required to be a member of this particular subset. This same set could be written $\{x : x \in \mathbf{R} \text{ and } x \geq 3\}$.

Exercise 1–4. Give this set a visual interpretation by graphing it on a number line.

Exercise 1–5. Translate into words: $\{x : x \in \mathbf{Q} \text{ and } x \leq 1/2\}$.

Exercise 1–6. Graph the set in the previous exercise. Is there a problem with making the graph?

Often, when the basic set is the real numbers, a short hand notation is used. One writes $[3, \infty)$ to denote $\{x \in \mathbf{R} : x \geq 3\}$. The conventions are these. If the endpoint is included in the set, a square bracket is used; otherwise a round bracket is used. The conceptual point infinity is almost never in the set, since infinity is not a real number.

Example 1–4. The set $\{x \in \mathbf{R} : x > 3\}$ could be written $(3, \infty)$, while $\{x \in \mathbf{R} : x \geq 3 \text{ and } x < 7\}$ could be written $[3, 7)$.

Problems

Problem 1–1. Find the set $\{x \in \mathbf{R} : \sqrt{x^2} = x\}$ and graph it.

Problem 1–2. Find the set $\{x \in \mathbf{R} : 2x + 3 = 5\}$ and graph it.

Problem 1–3. Find the set $\{x \in \mathbf{R} : (x + 2)^2 = x^2 + 4x + 4\}$ and graph it.

Problem 1–4. Find the set $\{x \in \mathbf{R} : \sqrt{x^2} = |x|\}$ and graph it.

Solutions to Problems

Problem 1–1. $\{x \in \mathbf{R} : \sqrt{x^2} = x\} = \{x \in \mathbf{R} : x \geq 0\}$. The graph is the half line beginning at the 0 and extending to the right.

Problem 1–2. $\{x \in \mathbf{R} : 2x + 3 = 5\} = \{x \in \mathbf{R} : x = 1\}$ which graphs as a single point.

Problem 1–3. $\{x \in \mathbf{R} : (x + 2)^2 = x^2 + 4x + 4\} = \mathbf{R}$. The equation $(x + 2)^2 = x^2 + 4x + 4$ is an example of an **identity**, since equality holds for all values of x for which both sides are defined.

Problem 1–4. This is another identity.

Solutions to Exercises

Exercise 1–1. The number $5.96 = 596/100$ is rational; $\sqrt{2}$ is not rational, but the proof is rather involved.

Exercise 1–2. Yes, it is $10/3$.

Exercise 1–3. Yes, rational numbers have decimal representations.

Exercise 1–5. The set of rational numbers that are less than or equal to $1/2$.

Exercise 1–6. In making the graph only the rational numbers should be shaded, but there are irrational numbers arbitrarily close to each rational number.

§2. Sets in Two Dimensions

In many cases the relationship of interest will be between two variables. A two dimensional set is used as the backdrop for visualizing this relationship.

In order to specify the values of two variables, two numbers are needed. The sets of the previous section depended on the properties of only a single variable. These sets were all subsets of the real numbers and could be represented visually on a line. For the new situation involving two variables the subsets will be visualized in a two dimensional plane. Denote by $\mathbf{R}^2$ the set $\{(x, y) : x \in \mathbf{R} \text{ and } y \in \mathbf{R}\}$. This is the set of ordered pairs (x, y) in which each member of the pair is a real number. The two numbers are called the **coordinates** of the point. Visually, $\mathbf{R}^2$ is a two dimensional plane. In order to assist in visualizing points in this plane, two coordinate axes are often drawn and the measuring units along each axis are marked.

Example 2–1. The set $\{(x, y) \in \mathbf{R}^2 : x = 3 \text{ and } y = 5\}$ can be visualized easily. This set consists of a single point. As a notational convenience, this point is written as $(3, 5)$. Caution: Do not confuse this notation with that used in the previous section for subsets of $\mathbf{R}$!

Example 2–2. A basic way of visualizing a more complicated set, such as $A = \{(x, y) \in \mathbf{R}^2 : y = 2x\}$ is to first rewrite this set as $\{(x, 2x) : x \in \mathbf{R}\}$ and then plot several individual points in the set, hoping to see a pattern. (This method is tedious for humans, but easy for computers.)

Exercise 2–1. What familiar geometric object is the set in the previous example?

Exercise 2–2. Is the set $B = \{(2x, 4x) : x \in \mathbf{R}\}$ the same as the set A?

The previous example suggests that sets in the plane are often specified by an equation which gives the required relationship between the two coordinates. This is indeed the case. Methods for obtaining this relationship will be discussed in the next section.

Once a set is visualized, it is possible to give at least an approximate answer to certain questions. Giving exact answers usually requires other methods.

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Problems

Problem 2–1. Write in set notation: the set of real numbers between 4 and 7, exclusive. What geometric object is this set?

Problem 2–2. Write in set notation: the set of points in the plane for which the second coordinate is 2 more than the first coordinate. What geometric object is this set?

Problem 2–3. Write the solid rectangular region with vertices at $(0,0)$, $(0,3)$, $(2,3)$, and $(2,0)$ in set notation.

Solutions to Problems

Problem 2–1. $\{x \in \mathbf{R} : 4 < x < 7\}$ or $(4, 7)$. This set is a line segment, without its endpoints.

Problem 2–2. $\{(x, y) \in \mathbf{R}^2 : y = x + 2\}$ or $\{(x, x + 2) : x \in \mathbf{R}\}$. This set is a line.

Problem 2–3. $\{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 3\}$.

Solutions to Exercises

Exercise 2–1. A line through the origin.

Exercise 2–2. Yes, each point in B has a second coordinate which is twice its first coordinate.

§3. Finding Relationships Between Variables

Relationships between variables are usually expressed in the form of equations. These equations typically arise in one or more of the following ways:

(1) from geometric reasoning applied to a picture of the relationship,

(2) from computing the same quantity in two different ways,

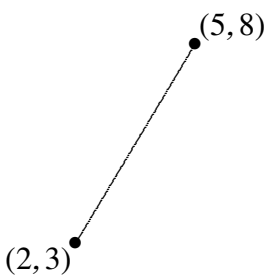
(3) from a verbal description of the relationship, or

(4) from an examination of experimental data.

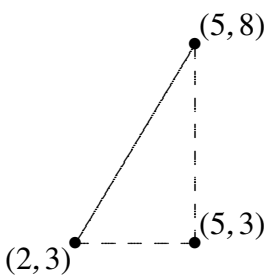
The last method will not be examined much in these notes, even though it is a very important method used in the experimental sciences. Some simple examples illustrating the other three methods are given in this section. These methods will appear repeatedly throughout the remainder of these notes.

The first examples illustrate the method of reasoning from a picture.

Example 3–1. What is the distance from $(2, 3)$ to $(5, 8)$? The picture below illustrates that the problem is to find the length of the line segment.



To do this, first add an extra point to the figure in order to create the right triangle shown below.



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The length of the horizontal side of the triangle is $5-2=3$; the length of the vertical side of the triangle is $8-3=5$. The Pythagorean Theorem then gives the length of the hypotenuse as $\sqrt{3^2+5^2}=\sqrt{34}$. This is the distance from $(2,3)$ to $(5,8)$.

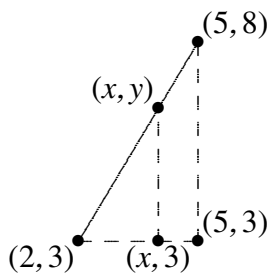
Exercise 3–1. What if the third point was chosen as $(2,8)$ instead of $(5,3)$?

The reasoning used in the previous example is completely general. The distance from the point (a,b) to (c,d) is $\sqrt{(a-c)^2+(b-d)^2}$.

Exercise 3–2. Draw an appropriate picture and establish this distance formula.

This same sort of picture reasoning can be used in other ways.

Example 3–2. What point is the midpoint of the line segment connecting $(2,3)$ and $(5,8)$? To answer this question, suppose (x,y) is the midpoint of the line segment. The problem is to find x and y. Add the point $(5,3)$ to the picture as before to create a right triangle and also indicate the midpoint (x,y) in the picture.



The point at the intersection of the vertical line segment from the (unknown) midpoint to the horizontal leg of the right triangle has coordinates $(x,3)$. The small and large right triangles are similar. So $(x-2)/(5-2)=1/2$ using the fact that the ratio of the length of the small hypotenuse to the length of the large hypotenuse is $1/2$. Hence $x=(2+5)/2$. Similarly, $y=(3+8)/2$, and the midpoint is $(7/2,11/2)$.

Exercise 3–3. Carefully fill in the details of how y was found.

Further examples of reasoning from pictures will be given in the remainder of these notes. This is an important technique to master.

The method of computing the same quantity in two different ways will be illustrated next.

Example 3–3. What is the equation of the line through the points $(2,3)$ and $(5,8)$? As stated, this question is a bit vague. The exact meaning is this. What relationship

must the coordinates (x, y) of a point satisfy if this point lies on the line through $(2, 3)$ and $(5, 8)$? The key fact in answering this question is that the slope of a line can be computed using any two points that lie on the line.

Exercise 3–4. Use a picture similar to that above to show that the slope can be computed using any two points on the line.

The slope of the line is now computed in two ways. First, since the points $(2, 3)$ and $(5, 8)$ lie on the line the slope is $\dfrac{8-3}{5-2} = 5/3$. Second, since the points $(2, 3)$ and (x, y) lie on the line the slope is $\dfrac{y-3}{x-2}$. Since these two values must be the same, the relationship $\dfrac{y-3}{x-2} = \dfrac{5}{3}$ must hold. Simplifying gives this relationship as $y = (5/3)x - (1/3)$ which must be satisfied by the coordinates of a point in order to lie on the line.

Exercise 3–5. What happens if the points $(5, 8)$ and (x, y) are used for the second computation?

Exercise 3–6. Describe the line in set notation.

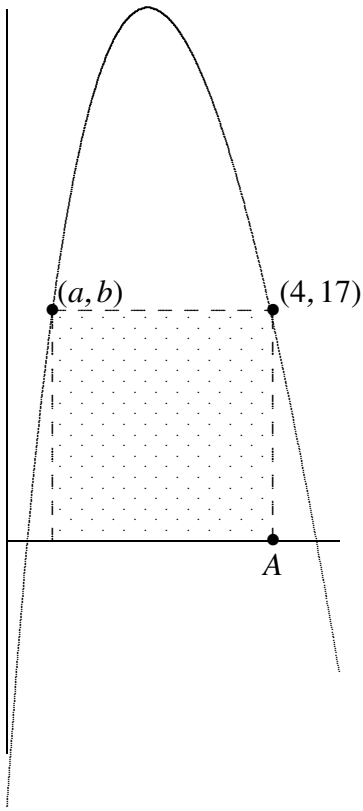
This section concludes with an example in which the equation relating the variables is obtained from a verbal description.

Example 3–4. The usual geometric description of a circle is as follows. A circle is the set of points in the plane which are a given distance, called the radius, from a fixed point in the plane called the center. Using this description, what is the equation that the coordinates (x, y) of a point must satisfy if that point lies on a circle of radius 5 with center at $(2, 3)$? From the first example above, the distance from (x, y) to $(2, 3)$ is $\sqrt{(x-2)^2 + (y-3)^2}$. From the verbal description, the point (x, y) is on the circle if and only if this distance is 5. The required relationship is therefore $\sqrt{(x-2)^2 + (y-3)^2} = 5$.

The relationships that exist between variables narrows the set under consideration from the whole plane (in the case of two variables) to a much smaller part of the plane. Finding the small number of points which are ultimately of interest is therefore made easier by exploiting these relationships.

Problems

Problem 3–1. These questions refer to the graph below in which a curve and the coordinate axes are shown. What are the coordinates of the point A? What is the value of b? What is the distance, in terms of a, from the point (a, b) to the point $(4, 17)$? Give an expression, in terms of a, for the area of the shaded region.



Problem 3–2. Find the equation of the line through the points $(-1, 2)$ and $(3, 7)$. Write the line as a set.

Problem 3–3. Find the equation of the line through the points $(3, 4)$ and $(3, 9)$. Write the line as a set.

Problem 3–4. Find the equation of the line through the point $(2, 1)$ which is parallel to the line $y = 7x - 4$. Write the line as a set.

Problem 3–5. What point is one-third of the way toward $(5, 8)$ from $(2, 3)$?

Problem 3–6. What equation must the coordinates (x, y) of a point D satisfy if the distance from D to the point $(2, 3)$ is 12?

Problem 3–7. What equation must the coordinates (x, y) of a point D satisfy if the distance from D to the point $(1, 2)$ is twice the distance from D to the point $(3, 4)$?

Problem 3–8. A parabola is the set of points in the plane which are equidistant from a given point, called the focus, and a given line, called the directrix. What equation must the coordinates (x, y) of a point satisfy if that point lies on the parabola with focus at $(3, 4)$ and directrix $y = -3$? Write the parabola as a set.

Problem 3–9. The line with equation $y = x$ is slid up 3 units and also 2 units to the right. What is the equation of the line in this new position? Hint: The point (x, y) lies on the new line if and only if the point $(x - 2, y - 3)$ lies on the old line.

Problem 3–10. The parabola with equation $y = x^2$ is slid up 4 units and 5 units to the left. What is the equation of the parabola in this new position?

Problem 3–11. When y is measured in pounds and x is measured in inches, the relationship between the two variables is $y = x^2$. What would be the relationship between x and y if y were measured in kilograms and x were measured in meters? (1 pound is 2.2 kilograms and 1 meter is 39.4 inches.)

Problem 3–12. What is the distance from the point with coordinates (a, b) to the line with equation $y = -x$?

Problem 3–13. What equation must the coordinates (x, y) of a point satisfy if that point lies on the parabola with focus at $(3, 4)$ and directrix $y = -x$?

Problem 3–14. An ellipse is the set of points in the plane the sum of whose distances from two given points in the plane, called the foci, is a given number. What is the equation that the coordinates of a point (x, y) must satisfy if the point is on the ellipse with foci at $(1, 3)$ and $(2, 7)$ and the given number is 9? Write the ellipse as a set.

Solutions to Problems

Problem 3–1. The coordinates of A are $(4, 0)$. Here $b = 17$. The distance from (a, b) to $(4, 17)$ is $4 - a$. The area is a rectangle with sides of length 17 and $4 - a$, so the area is $17(4 - a)$.

Problem 3–2. Computing the slope in two ways gives $\dfrac{2-7}{-1-3} = \dfrac{y-2}{x+1}$ as the equation. The line itself is the set $\{(x, y) \in \mathbf{R}^2 : y = (5/4)x + (13/4)\}$.

Problem 3–3. The equation is $x = 3$. What is the slope here?

Problem 3–4. Parallel lines have the same slope. Can you show this geometrically?

Problem 3–5. Draw a slightly different picture than in the midpoint formula case.

Problem 3–6. The distance formula gives $\sqrt{(x-2)^2 + (y-3)^2} = 12$.

Problem 3–7. The equation is $\sqrt{(x-1)^2 + (y-2)^2} = 2\sqrt{(x-3)^2 + (y-4)^2}$.

Problem 3–8. The distance from the point (x, y) on the parabola to the line $y = -3$ is $y + 3$.

Problem 3–9. $y - 3 = x - 2$.

Problem 3–10. The point (x, y) lies on the new parabola if and only if the point $(x + 5, y - 4)$ lies on the old parabola. The equation satisfied by points on the new parabola is therefore $y - 4 = (x + 5)^2$.

Problem 3–11. Here (x, y) is on the new curve if and only if $(39.4x, 2.2y)$ is on the old curve.

Problem 3–12. The point on the line with equation $y = -x$ closest to (a, b) has coordinates $(c, -c)$. The slope of the line through $(c, -c)$ and (a, b) is 1. Hence $b - (-c) = a - c$ and $c = (a - b)/2$. The distance from (a, b) to the line is $\sqrt{2}|a + b|/2$.

Problem 3–13. Apply the previous problem and the geometric definition of parabola.

Problem 3–14. $\sqrt{(x-1)^2 + (y-3)^2} + \sqrt{(x-2)^2 + (y-7)^2} = 9$.

Solutions to Exercises

Exercise 3–1. The orientation of the triangle changes, but the computed length is the same.

Exercise 3–3. Draw the horizontal line from the midpoint to the vertical leg of the right triangle. The new small triangle is similar to the original large one, so $(8-y)/(8-3) = 1/2$ and $y = 8 - 5/2 = 11/2$.

Exercise 3–5. In this case $(8-y)/(5-x) = 5/3$ and $y = (5/3)x - (1/3)$ as before.

Exercise 3–6. $\{(x, y) \in \mathbf{R}^2 : y = (5/3)x - (1/3)\}$ or equivalently, $\{(x, (5/3)x - (1/3)) : x \in \mathbf{R}\}$.

§4. Functional Relationships Between Variables

A functional relationship between variables exhibits the property that one would expect of a cause and effect relationship. Loosely, the variable y is a function of x if for each possible value of the variable x there is one and only one corresponding value of y.

Example 4–1. If the variables x and y are required to satisfy the equation $x^2 + y^2 = 1$ then there is not a functional relationship between the two variables. This is because, for instance, corresponding to the value $x = 0$ there are two values of y. The same argument also holds with the roles of x and y interchanged.

Exercise 4–1. If the variables x and y are required to satisfy the equation $x = y^2$ is there a functional relationship between x and y?

The preceding exercise points out the fact that it is entirely possible for x to be a function of y, but y to fail to be a function of x.

Exercise 4–2. What property of the graph of an equation involving two variables indicates the existence of a functional relationship?

To be precise, a function consists of three interrelated pieces:

(1) a set called the **domain** of the function,

(2) a set called the **range** of the function, and

(3) a rule which associates with each element of the domain exactly one element of the range.

Example 4–2. Suppose the domain and range are the set $\mathbf{R}$, and the rule is given by $x \mapsto x + 7$. (This rule is read "x maps into $x + 7$.") This trio does indeed define a function since there is no ambiguity about the element of the range any given element of the domain is associated with.

Exercise 4–3. What element of the range is associated with the element 4 of the domain? With the element -23?

Usually a name is given to the function and that same name is used notationally to express the rule.

Example 4–3. Suppose the name chosen for a function is g. Suppose also that the domain and range of g are the set $\mathbf{R}$. The rule would then be written $g(x) = x + 7$. The function g is then the same function as that of the previous example.

There is an important distinction which is often blurred. The function g consists

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of the three pieces: domain, range, and rule. The symbol $g(x)$ is the element of the range associated by the rule of the function g with the element x in the domain.

Example 4–4. The function h with domain and range $\mathbf{R}$ and rule $h(z) = z + 7$ is exactly the same function as g.

If a function f is known, a visual representation of f is given by the **graph** of f which is the set $\{(x, f(x)) : x \text{ is in the domain of } f\}$.

Example 4–5. For the function g above, the graph of g is a straight line.

Often, a function is specified by providing only the rule. In such cases the domain and range are given default values which are obtained as follows. The default domain is taken to be the largest set of elements to which the rule can be applied. Once the domain is determined, the range is the set of all values obtained by applying the rule successively to the elements of the domain.

Exercise 4–4. True or False: The range of a function can always be obtained from a knowledge of the domain and the rule.

Example 4–6. Suppose the rule for the function f is specified as $f(x) = \sqrt{x}$. The default domain is the answer to the question "for which values of x does $\sqrt{x}$ make sense?" The domain is therefore the interval $[0, \infty)$. The range is the answer to the question "what values can be written in the form $\sqrt{x}$ for some x in the domain?" The range is therefore $[0, \infty)$ too.

Because functional relationships are so important and common, the next several sections explore special types of functions that arise frequently.

Problems

Problem 4–1. True or False: For any real number x, $\sqrt{x^2} = x$.

Problem 4–2. What are the domain and range of the function h specified by the rule $h(x) = 1/x^2$?

Problem 4–3. What are the domain and range of the function g specified by the rule $g(x) = \dfrac{2x+4}{x^2-1}$?

Problem 4–4. The functions f and g are determined by the rules $f(x) = 2x^2 - 7x + 3$ and $g(w) = 2w^2 - 7w + 3$ respectively. Are the two functions the same?

Problem 4–5. The functions f and g are specified as follows. The function f has domain $[0, \infty)$ and rule $f(x) = 2x^2 - 7x + 3$; the function g has domain $[0, 1]$ and rule $g(w) = 2w^2 - 7w + 3$. Are the two functions the same?

Problem 4–6. The function H is given by the formula $H(z) = 2z^2 - 3$. Find the slope of the line through the two points $(1, H(1))$ and $(4, H(4))$ on the graph of H.

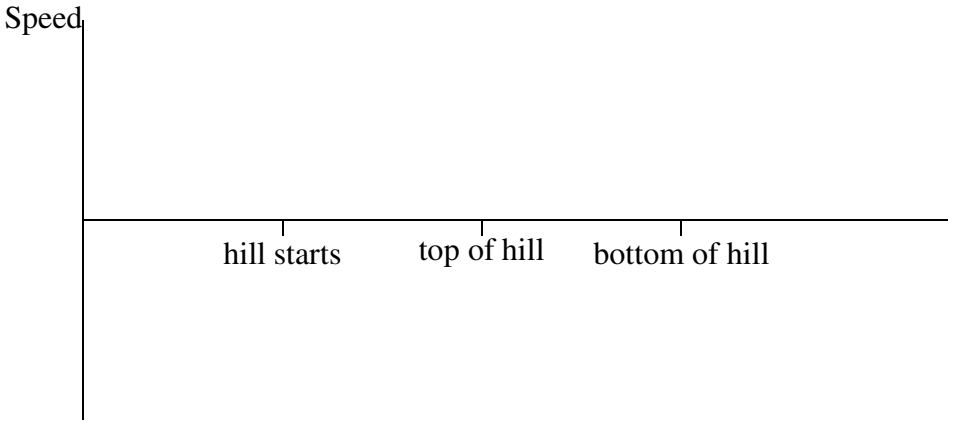
Problem 4–7. The total cost $C(q)$, in dollars, of manufacturing q radios is given by the formula $C(q) = 12q + 2700$. If production is increased by 10 from the current level, how much **additional** cost is incurred?

Problem 4–8. A water storage tank with a capacity of 100,000 gallons is initially empty and water will continuously enter the tank until it is full. Suppose $V(t)$ is the volume of water (in gallons) in the storage tank t hours after filling begins. Write an expression for the *percentage* change in the amount of water in the tank between the times 3 hours and 5 hours after filling begins. Write the equation you would solve in order to find the time at which the storage tank is half full.

Problem 4–9. The function D has as its rule that $D(h)$ is the density of ozone at an altitude of h meters above the ground. Write an expression for the difference in ozone density at the altitiudes of 500 and 1000 meters.

Problem 4–10. A bicyclist is riding along a level road at constant speed. After awhile, she approaches a small hill. To conserve energy, her speed decreases as she goes up the hill. At the top of the hill, she pedals energetically to increase her speed rapidly. After reaching the bottom of the hill, her speed decreases back to the

original steady level. On the axes below, plot her speed as a function of position.



Problem 4–11. The function S has domain $\mathbf{R}^2$ and range $\mathbf{R}^2$ and rule $S(x, y) = (x + 2, y + 4)$. Give a verbal description of this function. Inside what set would the graph of S be found?

Problem 4–12. The function M is defined by the rule $M(x, y) = x + y$. What are the domain and range of M? What familiar geometric object is the graph of M?

Solutions to Problems

Problem 4–1. False. The correct identity is $\sqrt{x^2} = |x|$.

Problem 4–2. The domain is the set of all real numbers except 0. The range is the set $(0, \infty)$.

Problem 4–3. To find the range, find the values of y for which the equation $y = \dfrac{2x+4}{x^2-1}$ has a solution x in the domain. This illustrates that although the range can always be found from the domain and the rule, the computations involved can be difficult.

Problem 4–4. Yes. The domains, ranges, and rules are the same. This means the functions are the same.

Problem 4–5. No. The functions have different domains. This alone is enough to make the functions different.

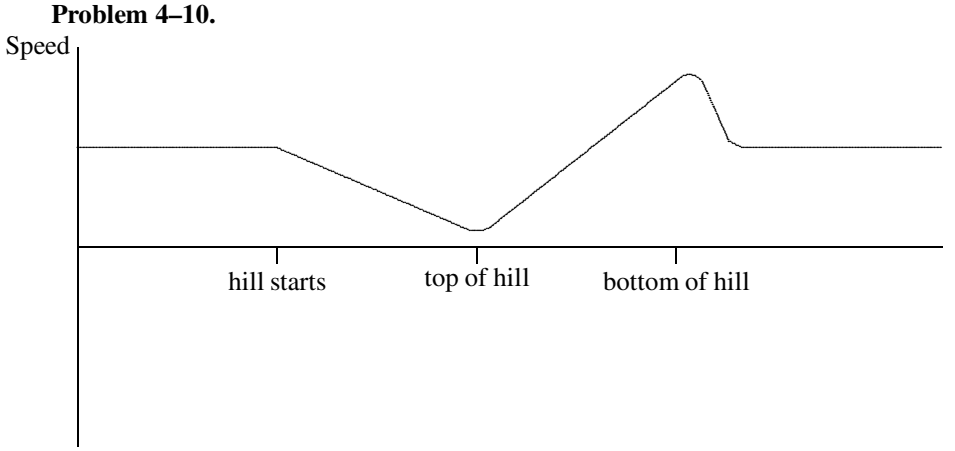
Problem 4–6. The coordinates of the two points in question are $(1, -1)$ and $(4, 29)$. The slope of the line through these two points is $(29 - (-1))/(4 - 1) = 30/3 = 10$.

Problem 4–7. If the current production level is x the current total cost is $12x + 2700$. The cost of producing $x + 10$ is $12(x + 10) + 2700$. Thus the additional cost is $12(x + 10) + 2700 - (12x + 2700) = 120$.

Problem 4–8. The percentage change is $100 \times \dfrac{V(5) - V(3)}{V(3)}$. The time t at which the tank is half full is the solution t of the equation $V(t) = 50,000$.

Problem 4–9. $D(500) - D(1000)$.

Problem 4–10.



Problem 4–11. The function S slides a point 2 units to the right and 4 units up. The graph of S would be in $\mathbf{R}^4$, the set of order 4tuples of real numbers.

Problem 4–12. The domain of M is $\mathbf{R}^2$, while the range is $\mathbf{R}$. The graph of M is 2 dimensional plane in 3 dimensional space.

Solutions to Exercises

Exercise 4–1. In this case y is not a function of x since there are two y values corresponding to each x value. But x is clearly a function of y. So there is a functional relationship between x and y.

Exercise 4–2. For y to be a function of x, each vertical line of the form $x = c$ must intersect the graph at most once; for x to be a function of y, each horizontal line of the form $y = c$ must intersect the graph at most once.

Exercise 4–3. 11 and -16 respectively.

Exercise 4–4. True, just apply the function to each element of the domain and keep track of the values that come out.

§5. Building New Functions From Old

Functions can be combined in simple ways to build up more complex functions. This sometimes allows the analysis of the behavior of a function to be broken into the analysis of these pieces.

The basic arithmetic operations provide simple ways of combining numbers. These same operations can be used to construct new functions from old. If f and g are functions, the function $f + g$ is defined by the rule $(f + g)(x) = f(x) + g(x)$; the function $f - g$ is defined by the rule $(f - g)(x) = f(x) - g(x)$; the function $f \cdot g$ is defined by the rule $(f \cdot g)(x) = f(x)\,g(x)$; the function f/g is defined by the rule $(f/g)(x) = f(x)/g(x)$.

Example 5–1. If $f(x) = x^2$ and $g(x) = \sqrt{x}$ then $(f + g)(3) = 9 + \sqrt{3}$, while $(f/g)(w) = w^2/\sqrt{w}$.

Exercise 5–1. What is $(f - g)(4)$? What is $(f \cdot g)(3)$?

A final way of combining functions is to apply the functions successively. This process is called **composition**. The composition of two functions f and g, denoted $f \circ g$, is defined by the rule $(f \circ g)(x) = f(g(x))$.

Example 5–2. In the previous example, $(f \circ g)(4) = f(g(4)) = f(2) = 4$.

Exercise 5–2. What is $(f \circ g)(z)$?

Being able to recognize the pieces from which a function is constructed is often valuable in calculus.

Example 5–3. The function $h(x) = (x^2 + 3)^7$ can be constructed from the simpler pieces $f(x) = x^2 + 3$ and $g(x) = x^7$, since $h = g \circ f$.

Exercise 5–3. Verify this assertion.

Exercise 5–4. From what simpler pieces can $\dfrac{1}{x^2 + 3}$ be constructed?

Some functions admit an "undo" rule. This allows the element in the domain to which the rule was applied to be determined from the element in the range produced by the rule.

Example 5–4. The function with rule $f(x) = x + 3$ has the undo rule $x \mapsto x - 3$.

Exercise 5–5. If $f(x) = 17$, what is x?

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Example 5–5. The function with rule $g(x) = x^2$ has no undo rule. This is because, for example, if $g(x) = 4$, there is no way to determine whether x was -2 or 2.

In order for a function to have an undo rule, the function must be **one-to-one**. This means that each element in the range is associated with one and only one element of the domain. If the function f has an undo rule, then the name of the function with the undo rule of f as its rule is called the **inverse function** of f, and is denoted by f^{-1}.

Example 5–6. For the function $f(x) = x + 3$, $f^{-1}(x) = x - 3$.

Example 5–7. If the function f has an inverse function f^{-1} then the domain of f^{-1} is the same as the range of f, while the range of f^{-1} is the same as the domain of f.

Inverse functions often have useful interpretations.

Example 5–8. Suppose $V(n)$ gives the ventilation requirements for n people to safely occupy a given room. Based on physical intuition, the function $V(n)$ should be increasing, and therefore have an inverse function. What is the physical interpretation of $V^{-1}(x)$? Evidently, this is the number of people who can safely occupy a room if x is the level of ventilation available.

Problems

Problem 5–1. Suppose $f(x) = 2x - 3$ and $g(x) = x^2 + 7$. What is $(f \cdot g)(x)$? What is $(f + g)(x)$? What is $(f \circ g)(x)$?

Problem 5–2. Does the function $h(x) = 1/x$ have an inverse function? If so, what is $h^{-1}(x)$?

Problem 5–3. True or False: If f is a function whose domain is all real numbers and f has an inverse function and if $f(3) = 19$ then $f^{-1}(19) = 3$.

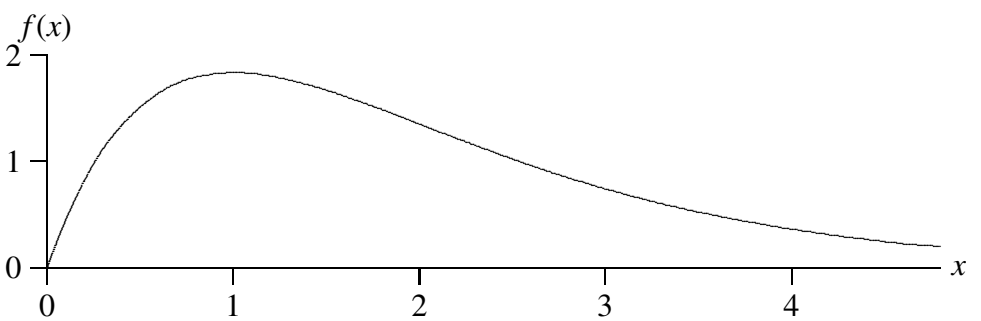
Problem 5–4. True or False: If g has an inverse function then $g^{-1}(g(x)) = x$ for all x in the domain of g.

Problem 5–5. True or False: If g has an inverse function, then so does g^{-1}, and the inverse function of g^{-1} is g.

Problem 5–6. Suppose h has an inverse function. If the point (x, y) lies on the graph of h^{-1} then on what graph does the point (y, x) lie?

Problem 5–7. Use the relationship in the previous problem to graph the inverse of the function $h(x) = \sqrt{x}$. (Hint: You can check your answer by finding a formula for h^{-1}.)

Problem 5–8. The questions here refer to the graph below of the function $f(x)$. How many solutions x are there to the equation $f(x) = 1$, which also satisfy $0 \le x \le 4$? Does the function $f(x)$ with domain $[0, 4]$ have an inverse function?



Solutions to Problems

Problem 5–1. $(f \cdot g)(x) = (2x - 3)(x^2 + 7)$, $(f + g)(x) = x^2 + 2x + 4$, $(f \circ g)(x) = 2(x^2 + 7) - 3$.

Problem 5–2. $h^{-1}(x) = 1/x$.

Problem 5–3. True.

Problem 5–4. True.

Problem 5–5. True.

Problem 5–6. If the point (x, y) lies on the graph of h^{-1} then the point (y, x) lies on the graph of h. This is because (x, y) lies on the graph of h^{-1} if and only if $y = h^{-1}(x)$, and this implies that $h(y) = x$, so the point (y, x) lies on the graph of h.

Problem 5–7. Here $h^{-1}(x) = x^2$ with domain $x \geq 0$.

Problem 5–8. There are two such solutions, one is between 0 and 1 and the other is between 2 and 3. They are located at the x coordinates of the intersection of the graph of $f(x)$ and the line $y = 1$. The graph fails the horizontal line test, so there is no inverse function.

Solutions to Exercises

Exercise 5–1. $(f-g)(4) = f(4) - g(4) = 16 - \sqrt{4} = 14$, and $(f \cdot g)(3) = f(3)g(3) = 9\sqrt{3}$.

Exercise 5–2. $(f \circ g)(z) = f(g(z)) = f(\sqrt{z}) = (\sqrt{z})^2 = z$.

Exercise 5–3. $(g \circ f)(x) = g(f(x)) = g(x^2 + 3) = (x^2 + 3)^7 = h(x)$.

Exercise 5–4. One choice involves the function $k(x) = 1/x$ and f from before.

Exercise 5–5. $x = 14 = 17 - 3$.

§6. Polynomials

Polynomials are a simple and useful class of functions. The three most important characteristics of polynomials are examined here. Understanding these characteristics allows an instant mental picture of the graph of any polynomial to be formed.

Linear and quadratic functions are both simple and useful. Somewhat more complicated functions can be built up in a simlar way. A **monomial** is a function with a rule of the form $x \mapsto cx^k$, where c is a constant and k is a non-negative integer.

Example 6–1. The functions with rules $x \mapsto 5x$, $x \mapsto 17x^2$ and $x \mapsto -3x^9$ are monomials while $x \mapsto \sqrt{x}$ and $x \mapsto x^{-3}$ are not.

A **polynomial** is a function whose rule is the sum of monomials. The **degree** of the polynomial is the power of largest power of the independent variable appearing in the polynomial.

Example 6–2. The linear and quadratic functions are polynomials of degree one and two respectively. The function with the rule $x \mapsto 3x^7 - 2x^3 + 3$ is a polynomial of degree 7. The function with rule $x \mapsto 13$ is a polynomial of degree 0.

A polynomial $p(x)$ of degree d has 3 essential characteristics:

(1) the value of $p(x)$ for large positive and negative values of x,

(2) the number of changes from increasing to decreasing, and vice-versa, and

(3) the number and location of the x intercepts (or roots) of $p(x)$, which are the values of x for which $p(x) = 0$.

Example 6–3. The polynomial $p(x) = 2x+7$ is of degree 1. If x is large and positive, then $p(x)$ is too; if x is large and negative, then $p(x)$ is too. This polynomial is always increasing and has one real root, namely $-7/2$. This simply means that $-7/2$ is the only solution of the equation $p(x) = 0$.

Exercise 6–1. True or False: The 3 characteristics of a polynomial of degree one depend only on the sign of the coefficient of x.

Exercise 6–2. Is every polynomial of degree 1 either always increasing or always decreasing?

Exercise 6–3. Where does the root appear on the graph of p?

Example 6–4. The polynomial $g(x) = x^2 + 8x - 12$ is of degree 2. If x is large and positive, then so is $g(x)$; if x is large and negative, $g(x)$ is large and positive. Notice

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that this depends only on the coefficient of x^2 being positive. Also, $g(x)$ has one change from decreasing to increasing. The polynomial $g(x)$ has two real roots.

Exercise 6–4. What are the two roots of $g(x)$?

Exercise 6–5. At which point does $g(x)$ change from decreasing to increasing?

Exercise 6–6. Does every polynomial of degree two have two real roots?

Exercise 6–7. Does every polynomial of degree two change from increasing to decreasing (or vice-versa) exactly once?

The Fundamental Theorem of Algebra states that a polynomial of degree d has d roots, provided that complex and multiple real roots are counted correctly.

Example 6–5. The polynomial $s(x) = x^2$ has a single real root of multiplicity 2.

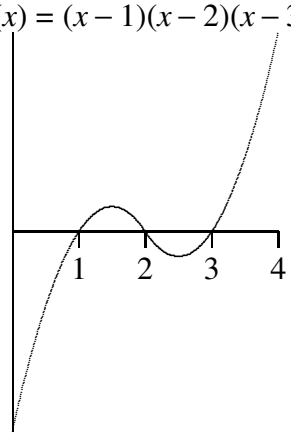
To illustrate the behavior when multiple real roots or complex roots are present 3 polynomials of degree 3 will be examined.

Example 6–6. The first is $f(x) = (x-1)(x-2)(x-3)$. For large positive values of x, $f(x)$ is large and positive; for large negative values of x, $f(x)$ is large and negative. This polynomial has 3 distinct x intercepts, at $x = 1$, $x = 2$, and $x = 3$. The graph of $f(x)$ is shown below. The function f changes from increasing to decreasing and vice-versa 2 times. This is the typical behavior of a polynomial of degree 3.

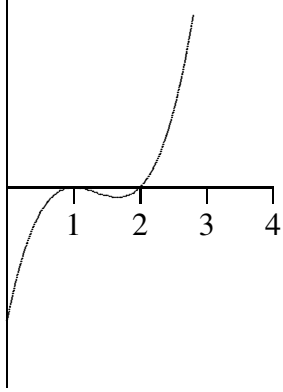
Example 6–7. The second polynomial is $g(x) = (x-1)^2(x-2)$. The behavior of g for large positive and negative values of x is the same as f. The repeated root at $x = 1$ causes the graph of g to remain below the axes until reaching $x = 2$. The graph of g also changes from increasing to decreasing and vice-versa 2 times.

Example 6–8. The third polynomial is $h(x) = (x-1)(x^2+1)$. The behavior of h for large positive and negative values of x is the same as for f and g. However h has only 1 x intercept (at $x = 1$) and is, in fact, always increasing.

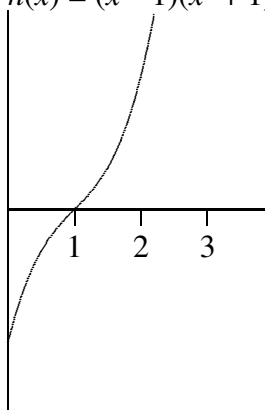
$f(x) = (x-1)(x-2)(x-3)$



$g(x) = (x-1)^2(x-2)$



$h(x) = (x-1)(x^2+1$



Generally, a polynomial of degree d will change from increasing to decreasing or vice-versa $d-1$ times. This will only fail to occur if the polynomial has repeated or complex roots.

Exercise 6–8. Give an example of a polynomial of odd degree d which is always increasing. Can the same be done for a polynomial of even degree?

The important facts about polynomials can be summarized as follows.

(1) The behavior of a polynomial for large positive and negative values of x is determined by the sign of the coefficient of the monomial of largest degree.

(2) The number of changes from increasing to decreasing or vice-versa can not exceed the degree minus one.

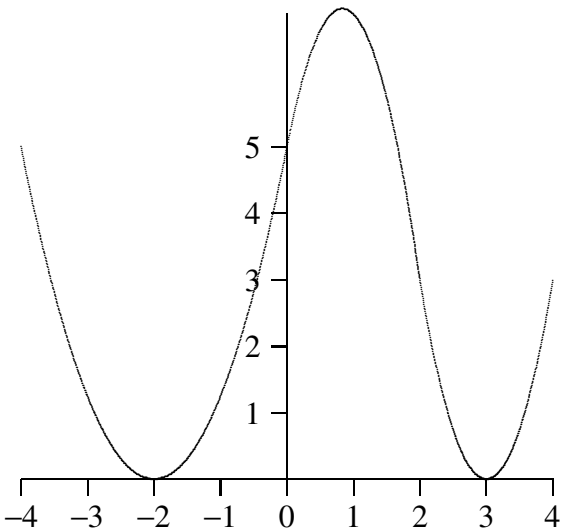
(3) The number of x intercepts can not exceed the degree.

By using these facts, a mental picture of the graph of any polynomial can instantly be obtained. Keep in mind that because of items (2) and (3), this picture may not be absolutely accurate. The picture does provide a good starting point for a more detailed analysis. This detailed analysis is more easily carried out by using calculus ideas.

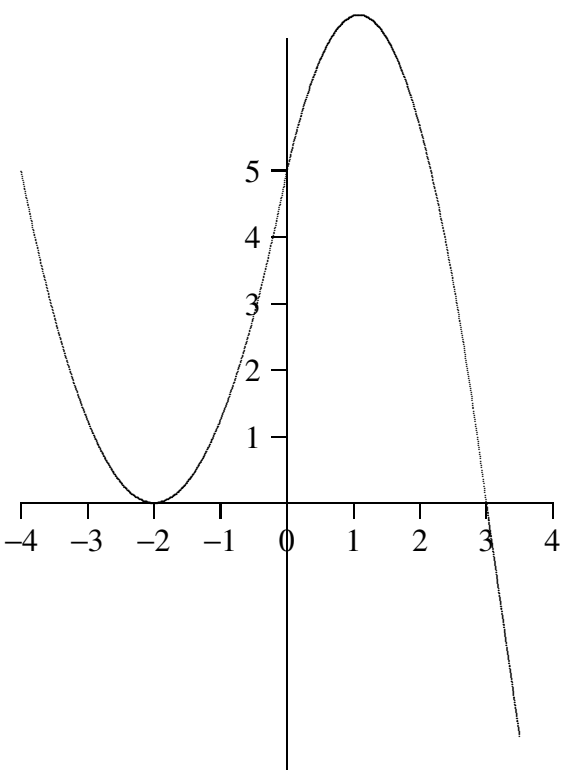
Problems

Problem 6–1. Give a quick sketch of the graph of $h(x) = 1 + 2x - 5x^2 + 4x^4$.

Problem 6–2. Find a possible formula for a polynomial whose graph is given by the following picture.



Problem 6–3. Find a possible formula for a polynomial whose graph is given by the following picture.



Problem 6–4. Is the sum of two polynomials a polynomial? Is the product of two polynomials a polynomial? Is the composition of two polynomials a polynomial? Is the quotient of two polynomials a polynomial?

Problem 6–5. The promoters of a rock concert believe that 30,000 tickets can be sold at a price of \$30 each. Their marketing survey indicates that for each \$5 increase in ticket price, 1,000 fewer tickets will be sold. Write an equation which gives the relationship between the number N of tickets sold and the price P per ticket. Write an equation which gives the relationship between the total ticket revenue R and the price P per ticket. At what price per ticket will total revenue be a maximum?

Problem 6–6. Find a quadratic polynomial which passes through the points $(0,0)$, $(1,1)$, and $(2,3)$. How many such polynomials are there? How many cubic polynomials pass through these same 3 points?

Problem 6–7. If f is a linear function with $f(2) = 5$ and $f(4) = 9$ then what is $f(x)$?

Problem 6–8. The owner of an ice cream franchise must pay the parent company \$500 per month plus 7% of her monthly gross sales S. Operating costs (rent, utilities, labor) of the franchise are \$1000 per month. The cost of raw materials for the ice cream is 50% of the gross sales. What is the equation giving the monthly profit P in terms of the gross monthly sales S?

Problem 6–9. Suppose that the cost of driving a car is a linear function of the number m of miles driven. Suppose that gas costs \$1.50 per gallon. Currently the car can travel 25 miles on a gallon of gas. If the car gets a tune up, which costs \$50, the car will be able to travel 30 miles on a gallon of gas. Approximately how many miles must the car be driven after a tune up to make the cost of a tune up worthwhile?

Problem 6–10. In most cases the roots of a polynomial can not be found by factoring. Suppose that by some means, perhaps a graph, two numbers $a < b$ can be found so that $f(a) < 0 < f(b)$. Argue that a root must lie between a and b. This is a special case of the **Intermediate Value Theorem:** a smooth function whose graph is below the axis at one point and above the axis at another point must cross the axis at some intermediate point.

Problem 6–11. The **bisection method** uses the fact of the previous problem to find a root to arbitrary accuracy. Let $m = (a+b)/2$ be the first coordinate of the midpoint of the line segment from $(a,0)$ to $(b,0)$. If $f(m) < 0$ then the root must lie between m and b; otherwise the root lies between a and m. The subdivision process is then repeated. Use the bisection method to find one root of $h(x) = 1 + 2x - 5x^2 + 4x^4$ to 3 decimal place accuracy.

Solutions to Problems

Problem 6–1. The sketch should be rising toward infinity for both large positive and negative values of x, and should change from decreasing to increasing to decreasing to increasing moving from left to right. The y intercept is at $(0, 1)$.

Problem 6–2. From its general shape, the polynomial should have degree 4. From the x intercepts, one possibility is $(x+2)^2(x-3)^2$. Since the y intercept is at $(0, 5)$, the final form could be $(5/36)(x+2)^2(x-3)^2$.

Problem 6–3. The behavior at $x = 3$ suggests one factor should be $(x-3)$ this time.

Problem 6–4. The answer is yes, except for quotients.

Problem 6–5. The information given implies that the relationship between N and P is linear. The (P, N) pairs $(30, 30000)$ and $(35, 29000)$ lie on the required line. The equation of this line is $\dfrac{30000-29000}{30-35} = \dfrac{N-30000}{P-30}$. This simplifies to $N = 36000-200P$. Since total revenue is the price per ticket times the number of tickets sold, $R = PN = P(36000-200P)$. This is a maximum when $P = 90$. How many tickets are sold at this price?

Problem 6–6. The candidate polynomial is $p(x) = ax^2 + bx + c$, where a, b, and c are to be determined. From the given information, $p(0) = 0$ so $c = 0$. Then from $p(1) = 1$, $a + b = 1$ while from $p(2) = 3$, $4a + 2b = 3$. From these last two equations a and b can be determined. There is only one quadratic polynomial which passes through these 3 points. There are infinitely many cubic polynomials passing through these same 3 points.

Problem 6–7. From the information given, the graph of f is a straight line. The slope of the line is $\dfrac{9-5}{4-2} = 2$. Thus $\dfrac{f(x)-9}{x-4} = 2$, or $f(x) = 2x+1$.

Problem 6–8. From the information given, $P = S-0.5S-1000-(500+0.07S)$, or $P = 0.43S-1500$.

Problem 6–9. The total cost to drive m miles without a tune up is $(1.50/25)m$ while the total cost to drive m miles after a tune up is $(1.50/30)m+50$. Equating these two expressions shows that the total cost to drive 5000 miles is the same.

Problem 6–10. The point $(a, f(a))$ is below the x-axis while the point $(b, f(b))$ is above the axis. The graph of f must cross the axis at least once between $(a, 0)$ and $(b, 0)$. This means that there is at least one root of f between a and b.

Problem 6–11. Here $h(-1) = -2$ and $h(0) = 1$ so there is at least one root between -1 and 0. At the first stage $m = -1/2$ and $h(-1/2) = -1$, so the root is between $-1/2$ and 0. At the second stage $m = -1/4$ and $h(-1/4) = 13/64$, so there is a root between $-1/2$ and $-1/4$. At the third stage $m = -3/8$ and $h(-3/8) = -383/1024$, so the root is between $-3/8$ and $-1/4$. Eventually, the root is -0.2961.

Solutions to Exercises

Exercise 6–1. True.

Exercise 6–2. Yes, it is increasing if the coefficient of x is positive and decreasing if the coefficient of x is negative.

Exercise 6–3. The real roots of the polynomial p appear as the x coordinates of the x intercepts.

Exercise 6–4. The equation $x^2 + 8x - 12 = 0$ has solutions $x = (8 \pm \sqrt{8^2 + 48})/2$ by the quadratic formula.

Exercise 6–5. At the vertex of the parabola, which is $x = -4$.

Exercise 6–6. No. The polynomial $x^2 + 1$ has no real roots, while x^2 has exactly one real root.

Exercise 6–7. Yes.

Exercise 6–8. The polynomials x and x^3 are always increasing. If the degree is even, this is not possible since the values must be large and positive (or large and negative) both when x is large and positive as well as when x is large and negative.

§7. Rational Functions

Polynomials can themselves be used as building blocks for more complicated functions. A function is a **rational function** if the function can be written as the ratio of two polynomials.

Example 7–1. The function $R(x) = \dfrac{2x^2 - 7x + 5}{x + 3}$ is a rational function, while $S(x) = \sqrt{x + 7}$ is not.

The behavior of a rational function $R(x)$ is characterized by

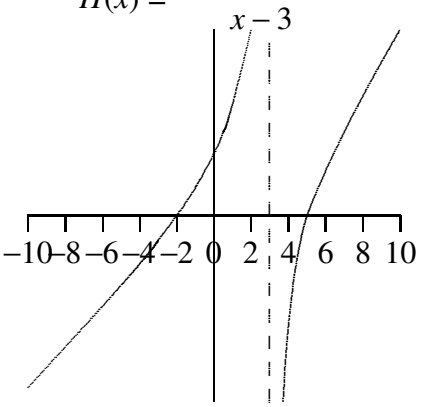
(1) the value of $R(x)$ for large positive and negative values of x,

(2) the number and location of the x intercepts of $R(x)$, and

(3) the behavior of $R(x)$ near points at which the denominator polynomial is zero.

Example 7–2. The behavior of $H(x) = \dfrac{(x+2)(x-5)}{x-3}$ can be studied as follows. For large positive or large negative values of x, $R(x)$ is about the same as $x^2/x = x$, and is large in the same way that x is large. The intercepts of H are also easily determined. The x intercepts depend only on the polynomial in the numerator of $H(x)$. The function H is not defined at $x = 3$, so the domain has a hole at $x = 3$. What is the behavior near $x = 3$? If x is slightly smaller than 3, the numerator is about -10 while the denominator is a small negative number. Thus is x is slightly smaller than 3, the value $H(x)$ is a large positive number. If x is slightly larger than 3, the numerator of H is still about -10 while the denominator is a small positive number. Thus $H(x)$ is a large positive number when x is slightly larger than 3. These considerations lead to the following rough picture.

$$H(x) = \frac{(x+2)(x-5)}{x-3}$$



The vertical line $x = 3$ is called a **vertical asymptote** (or **pole**) of the rational function H.

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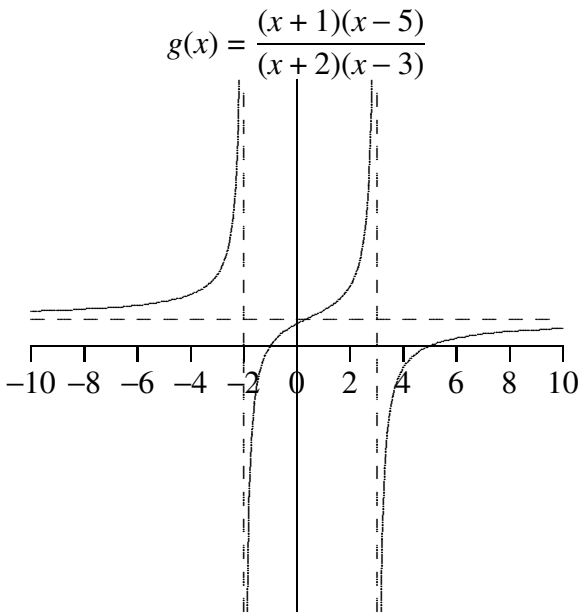
Exercise 7–1. What is the y intercept in the graph of H?

Exercise 7–2. Does every rational function have a vertical asymptote?

The possible existence of vertical asymptotes is not the only new feature that rational functions can display.

Example 7–3. The graph of the function $g(x) = \dfrac{(x+1)(x-5)}{(x+2)(x-3)}$ is shown below. For large positive or large negative values of x, the value of $g(x)$ is near $\dfrac{x^2}{x^2} = 1$. So in the extreme regions of the x axis, the graph must be near the horizontal line at height 1; the line $y = 1$ is called a **horizontal asymptote** in such cases. The x and y intercepts can be easily found in this case. Also, since the denominator vanishes at $x = -2$ and $x = 3$ there will be two vertical asymptotes.



Exercise 7–3. Where is the y intercept of the function g?

Rational functions can exhibit two new kinds of behavior that polynomials can not: the existence of horizontal asymptotes and vertical asymptotes. These properties enable rational functions to serve as models of physical phenomena for which a polynomial model would be unreasonable.

Example 7–4. Suppose $R(c)$ is the rate at which alcohol is removed from the bloodstream when the blood alcohol concentration is c. Since liver function in removing alcohol is limited, $R(c)$ should be a function with a horizontal asymptote.

Exercise 7–4. Where should the x and y intercept(s) of R be located?

Problems

Problem 7–1. True or False: The function $f(x) = \dfrac{\sqrt{x^2+3}}{3x^2-7x}$ is a rational function.

Problem 7–2. Does every rational function have a horizontal asymptote?

Problem 7–3. The graph of a rational function has a horizontal asymptote at $y = 3$ and vertical asymptotes at $x = -3$ and $x = 6$. Find a possible formula for such a rational function.

Problem 7–4. Suppose $D(t)$ is the distance travelled by a motor boat in t seconds after the failure of the motor. The distance is measured from the point at which the motor failed. Sketch the graph of $D(t)$.

Problem 7–5. A rectangular house is to be built with exterior walls that are 8 feet high. One wall of the house will face north. The total enclosed area of the house will be 1500 square feet. Annual heating costs for the house are determined as follows. Each square foot of exterior wall with a northern exposure adds \$4 to the annual heating cost; each square foot of exterior wall with an eastern or western exposure adds \$2 to the annual heating cost; each square foot of exterior wall with a southern exposure adds \$1 to the annual heating cost. Denote by L the length of each of the north and south facing walls, and by W the length of each of the east and west facing walls. Write an equation that expresses the total annual heating cost C in terms of L. For which values of L is this formula valid? For approximately what value of L is the annual heating cost minimized, and what is the annual heating cost for this choice of L?

Solutions to Problems

Problem 7–1. False. A rational function is the ratio of two polynomials, and the numerator is not a polynomial.

Problem 7–2. The first example of this section shows that the answer is no. A horizontal asymptote will exist if and only if the degree of the denominator polynomial is at least as large as the degree of the numerator polynomial.

Problem 7–3. One choice would be $\dfrac{3x^2}{(x+3)(x-6)}$. There are many others.

Problem 7–4. The graph should have $D(0) = 0$ and the graph should have a horizontal asymptote. The level at which the asymptote appears corresponds to the maximum distance travelled by the boat after engine failure.

Problem 7–5. From the information given, $C = 8(4L + 2W + 2W + L) = 40L + 32W$. Since the area of the house is 1500 square feet, $LW = 1500$ so $W = 1500/L$. Hence $C = 40L + 32 \times 1500/L$. This formula is valid for $L > 0$. By making a reasonably careful graph of C versus L, the annual heating costs are minimized at about $L = 35$, which corresponds to an annual heating cost of about \$2900.

Solutions to Exercises

Exercise 7–1. The y intercept is located at $(0, H(0)) = (0, 10/3)$.

Exercise 7–2. No. The rational function $1/(1 + x^2)$ has no vertical asymptote.

Exercise 7–3. At $(0, g(0)) = (0, 5/6)$.

Exercise 7–4. Evidently, $R(0) = 0$ so $(0, 0)$ should be the only x and y intercept.

§8. Finding a Function from Its Properties

In applications the properties of a function are often specified instead of directly specifying the function itself. Usually this specification can be turned into a conventional formula after some additional work.

Example 8–1. You deposit \$100 into a bank account which pays 5% interest compounded annually. If no additional deposits are made other than interest, what is the amount $A(t)$ of money in the account t years after the initial deposit? In this context, the unknown function $A(t)$ is not given explicitly, but by using the verbal interpretation of $A(t)$ it is possible to find the relationship between $A(t+1)$ and $A(t)$. This is because $A(t+1)$ is the amount in the account $t+1$ years after the initial deposit, and this must differ from the amount in the account one year earlier only by the amount of interest received in that year. Hence

$$A(t+1) = A(t) + 0.05A(t) = (1.05)A(t).$$

This relationship holds for any value of t. Now by using specific values of t, the form of the function $A(t)$ can be found. Here $A(0) = 100$, and using this relation with $t = 0$ gives $A(1) = (1.05)A(0) = 100(1.05)$. Then using the relation with $t = 1$ gives $A(2) = (1.05)A(1) = 100(1.05)^2$, after using the previous fact. Continuing in this way leads to the general formula $A(t) = 100(1.05)^t$.

Exercise 8–1. How would the example change if the interest rate was 5% compounded quarterly?

The function $A(t)$ in the example exhibits a different form than the functions considered earlier. A function with rule of the form $x \mapsto r^x$ for some positive number r is called an **exponential function**. Exponential functions are among the most important functions in mathematics. This is because exponential functions arise in the solution of many varied applied problems.

Example 8–2. Experments have shown that the intensity of light decreases by a factor of 20% for each half meter of depth below the surface of a lake. What is the intensity $I(d)$ at a depth of d meters? From the information given, $I(d+1/2) = 0.8I(d)$ for any depth d. From here it is easy to find the intensity at any depth in terms of the intensity at the surface.

Exercise 8–2. What is the formula for $I(d)$?

Exponential functions increase and decrease more rapidly than any of the other functions studied so far. To see this, compute the percentage change in the value of the exponential function r^x over any interval of length 1. Using properties of

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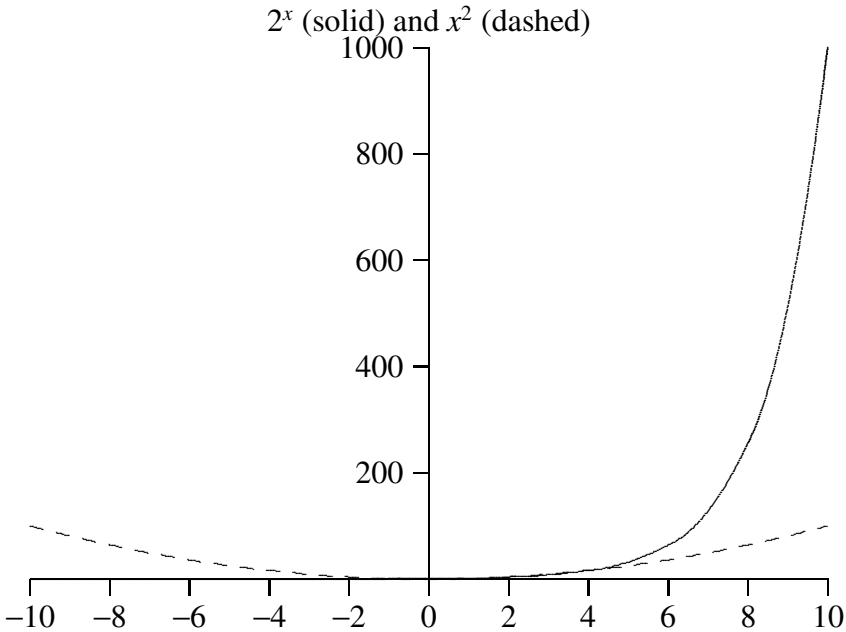
exponents, if the interval is $[a, a+1]$ then the percentage increase is $100\times\dfrac{r^{a+1}-r^a}{r^a} = 100(r-1)$ percent. Since this percentage increase does not depend on the location of the interval, an exponential function will increase (or decrease) by the same percentage over any interval of length 1. Polynomials and rational functions increase by a decreasing percentage over intervals of fixed length. So exponential functions increase or decrease much faster than either polynomials or rational functions. This is illustrated in the graph below.



Exercise 8–3. Compute the percentage increase of the function $p(x) = x^2$ over the intervals $[1, 2]$, $[10, 11]$, and $[100, 101]$ and compare these with the percentage increase of the function 2^x over the same intervals.

The different exponential functions are in fact closely related.

Example 8–3. How do the graphs of 2^x and 3^x differ? Looking at the graph of 2^x shows that there is a number, call it b, so that $2^b = 3$. From this, $3^x = (2^b)^x = 2^{bx}$ for all numbers x. Thus the graph of 3^x is just a horizontally compressed version of the graph of 2^x.

Using the same idea as in the previous example, any one exponential function can be easily expressed in terms of any other exponential function. For reasons that can only be explained using calculus ideas, there is one exponential function which is of primary interest. The number $e = 2.71828\ldots$ is the base for this exponential function. The exponential function e^x is called the **natural exponential function**.

Exercise 8–4. What is the graph of e^{-x}?

Because of the close connections between exponential functions and applications, the exponential functions are probably the most important functions in mathematics.

Problems

Problem 8–1. True or False: For any real numbers x and y, $e^{x+y} = e^x e^y$.

Problem 8–2. True or False: For any real numbers x and y, $(e^x)^y = e^{xy}$.

Problem 8–3. Sketch the graph of the function $\dfrac{e^x}{1+e^x}$. What are the horizontal asymptotes?

Problem 8–4. In marine biology the photonic zone is the top layer of the ocean, and ends at the depth at which 1% of the light penetrates. In the Caribbean, 50% of the light reaches a depth of 13 meters. In New York harbor, 50% of the light does not reach a depth of 10 centimeters. What is the depth of the photonic zone in the Caribbean? In New York harbor?

Problem 8–5. Suppose the function A is defined by the rule that $A(c)$ is the total cost of producing c cars. How can the additional cost of increasing production from 500 cars to 600 cars be written how in terms of the function A?

Problem 8–6. You are observing a population of beetles in a laboratory environment. The time at which you begin your observation is time $t = 0$. Denote by $B(t)$ the number of female beetles in the population at the end of t weeks of observation. Assume that female beetles never die, and that each female beetle that is at least 2 weeks old gives birth to 3 female beetles each week. Assume that births occur just before the weekly population count. At time $t = 0$ there are exactly 4 female beetles in the population, and they are all newborns. Hence $B(0) = 4$. Compute $B(1)$, $B(2)$, and $B(3)$. Write the equation that expresses the relationship between $B(t+2)$, $B(t+1)$, and $B(t)$ which holds for $t \geq 0$.

Problem 8–7. Denote by $A(t)$ the amount of radioactive material that remains after t years when beginning with the amount $A(0)$. Current models for radioactive decay indicate that the percentage of the radioactive substance that decays in any one year period is a constant c which depends only on the type of radioactive substance. Find a formula for $A(t)$.

Problem 8–8. Newton's law of cooling states that the rate of cooling of a body immersed in a bath is proportional to the temperature difference between the body and the bath. Suppose a body with an initial temperature of 100 degrees is immersed in a bath which retains a constant temperature of 30 degrees. Denote by $T(t)$ the temperature of the body after t minutes. Find the relationship between $T(t+1)$ and $T(t)$.

Problem 8–9. A person is taking a dose of a drug at regular time intervals. Denote by $C(n)$ the concentration of the drug in the bloodstream immediately after the nth

dose is taken. Assume that between doses the body metabolizes 60% of the drug in the bloodstream, and that each dose immediately boosts the concentration of the drug in the bloodstream by 0.1 units. What is the relationship between $C(n+1)$ and $C(n)$? What is the approximate concentration of the drug in the bloodstream after many doses have been taken?

Problem 8–10. The exponential function e^x is an increasing function and therefore has an inverse function. This inverse function is called the **natural logarithm** function and is denoted $\ln x$. What is the domain and range of the natural logarithm function?

Problem 8–11. True or False: If x and y are positive numbers then $\ln(xy) = \ln(x) + \ln(y)$.

Problem 8–12. True or False: If x is a positive number and y is any number then $\ln(x^y) = y \ln x$.

Problem 8–13. Solve the equation $e^{3x} = 7$ for x.

Problem 8–14. The engine of a speeding motorboat fails, and the drag force of the water causes the boat to come to a stop. Suppose time $t = 0$ corresponds to the instant that the motor failed. The distance $D(t)$ in meters that the boat travels in t seconds, measured from the point at which the motor failed, is given by $D(t) = 100(1 - e^{-t/5})$. How many seconds does it take for the boat to travel 50 meters from the point at which the motor failed? How far does the boat ultimately travel from the point at which the motor failed? (Hint: You may wish to make a rough sketch of the graph of $D(t)$.)

Solutions to Problems

Problem 8–1. True.

Problem 8–2. True.

Problem 8–3. The horizontal asymptote to the left is $y = 0$, while the horizontal asymptote to the right is $y = 1$.

Problem 8–4. If $I(d)$ is the light intensity at a depth of d meters, the information for the Caribbean gives $I(d + 13) = 0.5I(d)$, assuming the light conduction properties of the water do not change with depth. Hence $I(d) = I(0)(0.5)^{d/13}$, and $I(d)/I(0) = 0.01$ when $d = 13\ln(.01)/\ln(0.5) = 86.37$. A similar argument applies for New York harbor.

Problem 8–5. The additional cost is $A(600) - A(500)$.

Problem 8–6. From the information given, $B(1) = 4$, $B(2) = 4 + 4 \times 3 = 16$, and $B(3) = 16 + 4 \times 3 = 28$. Generally, $B(t + 2) = B(t + 1) + 3B(t)$.

Problem 8–7. From the information given $100\dfrac{A(t+1) - A(t)}{A(t)} = 100 - c$, and after re-arranging, $A(t+1) = (1 - c/100)A(t)$. From here, $A(t) = (1 - c/100)^t A(0)$.

Problem 8–8. The (average) rate at which the temperature of the body is changing in any one minute is $\dfrac{T(t+1) - T(t)}{1}$.

Problem 8–9. $C(n + 1) = C(n) - 0.6C(n) + 0.1$. The horizontal asymptote of the graph of $C(n)$ will answer the second question. Where is this asymptote?

Problem 8–10. The domain of the exponential function is the set **R** of all real numbers, so this is the range of the logarithm. The range of the exponential function is the set of positive real numbers, so this is the domain of the logarithm.

Problem 8–11. True. This follows from the fact that for any numbers a and b, $\ln(e^{a+b}) = a + b = \ln(e^a e^b)$ together with the fact that if x and y are positive then $x = e^a$ and $y = e^b$ for some numbers a and b.

Problem 8–12. True. Can you give an explanation?

Problem 8–13. Since e^x and $\ln x$ are inverse functions, $\ln(e^{3x}) = 3x$. Hence the equation $e^{3x} = 7$ holds if and only if $3x = \ln 7$, or $x = \ln 7/3$.

Problem 8–14. The time t required to travel 50 meters is the solution of the equation $50 = 100(1 - e^{-t/5})$. Hence $t = -5\ln(1/2) = 3.465$ seconds. For large positive values of t, $D(t)$ is about 100. Hence the boat travels about 100 meters from the point at which the motor failed. In the graph of $D(t)$, the horizontal asymptote of D is at 100.

Solutions to Exercises

Exercise 8–1. The relationship is then $A(t + 1/4) = (1 + 0.05/4)A(t)$.

Exercise 8–2. Here $I(d) = (0.8)^{2d}I(0)$.

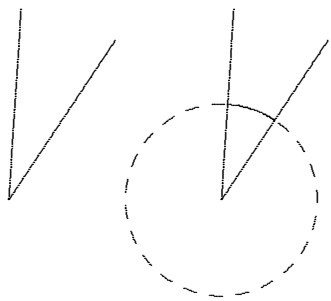
Exercise 8–3. The percentage increase of the function p are 300%, 21%, and 2.01% respectively, while 2^x increases by 300% over each of the intervals.

Exercise 8–4. The graph of e^{-x} decreases rapidly from left to right. To see the picture accurately, just view the graph of e^x from the backside of the page!

§9. Trigonometric Relationships

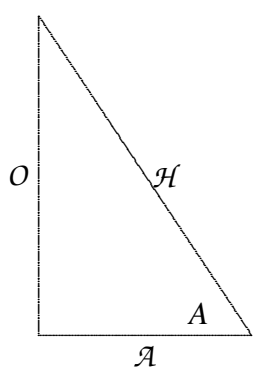
None of the functions studied so far have the repeating property that would be expected in modeling a physical phenomenon such as ocean waves. Amazingly, the functions required in such models have their origin in the study of triangles.

An **angle** is formed by two rays emanating from a common point, called the **vertex** of the angle. In order to measure the size of an angle, a simple geometric construction is used. Construct a circle of any convenient radius r with center at the vertex of the angle. Measure the length s of the arc intercepted by the angle on the circumference of the circle. In the picture below the intercepted arc is the solid part of the circle. The radian measure of the angle is then defined to be s/r. Geometric considerations show that this ratio depends only on the angle and not on the choice of radius of the circle.



Exercise 9–1. What is the radian measure of a right angle?

Because of the side-side-side congruence theorem, the angles of the triangle are completely determined once the lengths of the sides are known. This means that functions of the angles of the triangle can be well defined in terms of the lengths of the sides. In the right triangle shown below, the length of the side opposite the angle A has length $\mathcal{O}$, the length of the side adjacent to the angle A has length $\mathcal{A}$, and the hypotenuse has length $\mathcal{H}$.



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The sine and cosine functions are defined by the formulas

$$\sin A = \frac{\mathcal{O}}{\mathcal{H}} \qquad\qquad \cos A = \frac{\mathcal{A}}{\mathcal{H}}.$$

Because the lengths of the sides of similar triangles are proportional, these definitions depend only on the measure of the angle A and not on the particular right triangle in which A appears.

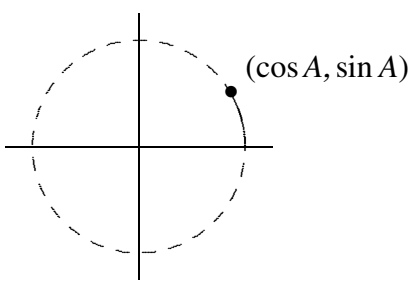
Example 9–1. The acute angles of an isoceles right triagle each have measure $\pi/4$. Since the length of the hypotenuse of such a triangle is $\sqrt{2}$ times the length of the sides, $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$.

Example 9–2. Begin with an equilateral triangle. The angles in this triangle must each have measure $\pi/3$. Draw the perpendicular bisector of one of the angles and consider one of the two right triangles formed by doing so. Applying the definition of sine and cosine in this right triangle gives $\sin(\pi/6) = 1/2$, $\cos(\pi/6) = \sqrt{3}/2$, $\sin(\pi/3) = \sqrt{3}/2$, and $\cos(\pi/3) = 1/2$.

Exercise 9–2. Fill in the details of the preceding example.

As defined so far, the sine and cosine functions have as their domain the angles with radian measure between 0 and $\pi/2$, exclusive of the endpoints. Certainly such functions could not provide a model for the oscillatory behavior of ocean waves! The key to expanding the domain of these functions is to return to the way in which angles are measured.

Suppose that the circle of radius 1 with center at the origin is drawn. An arc of this unit circle of length A is marked off beginning from the point $(1, 0)$. If A is positive, the arc is marked off in the counter-clockwise direction; if A is negative the arc is marked off in the clockwise direction. The corresponding point on the circle is defined to have the coordinates $(\cos A, \sin A)$. This process is illustrated in the picture below.



This definition extends the domain of the sine and cosine functions to the set of all real numbers. By constructing the right triangle with vertices at the origin, $(\cos A, \sin A)$ and $(\cos A, 0)$, it is easy to see that the new definitions of sine and cosine agree with the previous ones for angles between 0 and $\pi/2$.

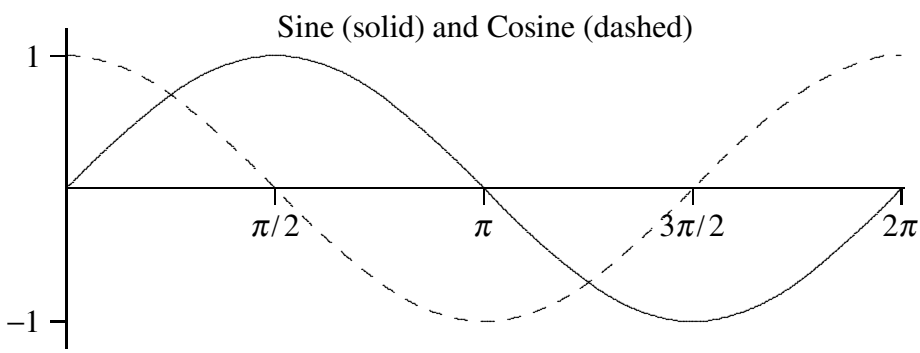
Example 9–3. Since by definition the point $(\cos A, \sin A)$ lies on the unit circle, the identity $(\cos A)^2 + (\sin A)^2 = 1$ holds for any real number A. This is called the **Pythagorean Identity**.

Example 9–4. The special shorthand notation $(\cos A)^2 = \cos^2 A$ and $(\sin A)^2 = \sin^2 A$ is often used. A similar notation can be used with the other trigonometic functions.

Example 9–5. For some angles the values of sine and cosine can be easily computed. When marking off the angle $\pi/2$ the terminal point is $(0, 1)$. This means that $\cos \pi/2 = 0$ and $\sin \pi/2 = 1$.

Exercise 9–3. What is $\cos \pi$? What is $\sin \pi$? What is $\cos(3\pi/2)$? What is $\sin(3\pi/2)$?

With this information in mind, the graphs of the sine and cosine functions can be roughly drawn.



Exercise 9–4. The picture suggests that the graph of the cosine function can be obtained by sliding the graph of the sine function to the left. Use this idea to show that $\sin x = \cos(x - \pi/2)$ for all x.

Some other trigonometric functions are defined in terms of sine and cosine. The most important one is the tangent function, which is defined as $\tan x = \dfrac{\sin x}{\cos x}$.

Exercise 9–5. If A is the angle in the right triangle above, what is $\tan A$ in terms of the lengths of the sides of the triangle?

The trigonometric functions of lesser importance are the secant (defined by $\sec x = 1/\cos x$), the cosecant (defined by $\csc x = 1/\sin x$), and the cotangent (defined by $\cot x = 1/\tan x$). These functions occasionally appear in formulas. Note that in all cases these functions are nothing more than abbreviations for certain ratios involving sine and/or cosine.

Problems

Problem 9–1. True or False: For any angle A, $\cos A = \cos(-A)$.

Problem 9–2. True or False: For any angle A, $\sin A = -\sin(-A)$.

Problem 9–3. What is the relationship between $\sin x$ and $\sin(\pi - x)$? Between $\cos x$ and $\cos(x - \pi)$?

Problem 9–4. Sketch the graph of the tangent function. What is the domain of this function? What is the range of this function?

Problem 9–5. True or False: For any angle x in the domain of the tangent function, $\tan(x + \pi) = \tan x$.

Problem 9–6. In view of the preceding problems, the tangent function (as it has been defined) does not have an inverse function. However, on the interval $(-\pi/2, \pi/2)$ the tangent function is increasing. The function with domain $(-\pi/2, \pi/2)$ and the same rule as the tangent function does have an inverse function, and this inverse function is called $\arctan x$ (sometimes denoted $\tan^{-1} x$). For which values of x do the relations $\tan(\arctan x) = x$ and $\arctan(\tan x) = x$ hold? What is $\cos(\arctan 5)$? What is $\cos(\arctan x)$?

Problem 9–7. Find all solutions of the equation $\tan x = 1$.

Problem 9–8. The new function with the domain $[-\pi/2, \pi/2]$ and the same rule as the sine function also has an inverse function which is $\arcsin x$. The new function with domain $[0, \pi]$ and the same rule as the cosine function has an inverse function which is $\arccos x$. What are the domain and range of $\arcsin x$? What are the domain and range of $\arccos x$? For which values of x do the identities $\sin(\arcsin x) = x$, $\arcsin(\sin x) = x$, $\cos(\arccos x) = x$, and $\arccos(\cos x) = x$ hold?

Problem 9–9. Find all solutions of the equation $\sin x = 1/2$.

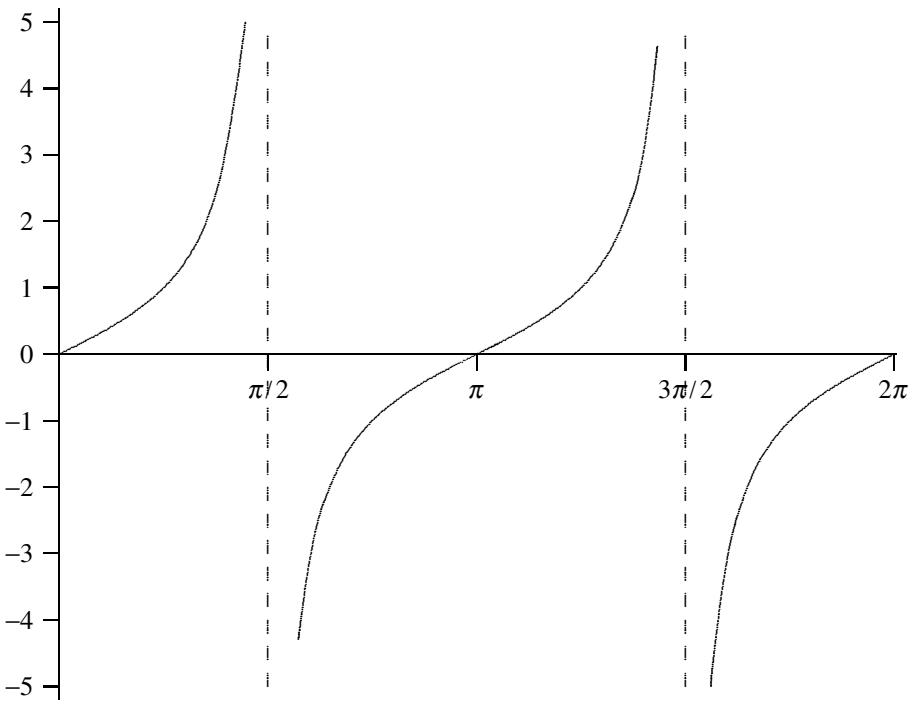
Solutions to Problems

Problem 9–1. True. Mark off the angles A and $-A$ on the unit circle and observe by symmetry that the points $(\cos A, \sin A)$ and $(\cos(-A), \sin(-A))$ lie on a vertical line.

Problem 9–2. True, using the same picture as in the previous problem.

Problem 9–3. Marking off the angles x and $\pi - x$ on the unit circle shows that $\sin x = \sin(\pi - x)$, while $\cos x = -\cos(\pi - x) = -\cos(x - \pi)$.

Problem 9–4.



The domain of $\tan x$ is all real numbers except for $\ldots, -\pi/2, \pi/2, 3\pi/2, \ldots$. The range of the tangent function is all real numbers.

Problem 9–5. True. This is because $\sin(x+\pi) = -\sin x$ and $\cos(x+\pi) = -\cos x$. Can you justify these two identities?

Problem 9–6. The identity $\tan(\arctan x) = x$ is true for all real numbers x, and $\arctan(\tan x) = x$ holds for $-\pi/2 < x < \pi/2$. Since $\arctan 5$ is an angle between 0 and $\pi/2$, its cosine can be computed using a right triangle. Hence $\cos(\arctan 5) = 1/\sqrt{26}$. Generally, $\cos(\arctan x) = 1/\sqrt{1 + x^2}$.

Problem 9–7. The basic solution is $x = \arctan 1 = \pi/4$. Since $\tan(x+\pi) = \tan x$ for all x, other solutions can be obtained by adding an integer multiple of π to this basic solution. The general form of the solutions is therefore $\pi/4 + k\pi$, where k is an integer. To visualize this process, there are two possible pictures. One graph has the unit circle and the line $y = x$ drawn in it; the intersection of the line and the circle mark the basic solutions. The other graph contains the

graph of $\tan x$ and the line $y = 1$; the intersection of the line and the graph of $\tan x$ provide all solutions.

Problem 9–8. The domain of $\arcsin x$ is $[-1, 1]$ and the range of $\arcsin x$ is $[-\pi/2, \pi/2]$. The domain of $\arccos x$ is $[-1, 1]$ and the range of $\arccos x$ is $[0, \pi]$. The identity $\sin(\arcsin x) = x$ holds for $-1 \le x \le 1$; the identity $\arcsin(\sin x) = x$ holds for $-\pi/2 \le x \le \pi/2$; the identity $\cos(\arccos x) = x$ holds for $-1 \le x \le 1$; the identity $\arccos(\cos x) = x$ holds for $0 \le x \le \pi$.

Problem 9–9. The basic solution is $x = \arcsin(1/2) = \pi/6$. Another solution is $\pi - \pi/6$. All other solutions are obtained by adding integer multiples of 2π to these. Can you give two different graphs which interpret this process?

Solutions to Exercises

Exercise 9–1. The intercepted arc is a quarter circle; hence the measure of a right angle is $(2\pi r/4)/r = \pi/2$.

Exercise 9–2. If each side of the equilateral triangle has length L, then the hypotenuse of the right triangle has length L also. The two legs of the right triangle have lengths $L/2$ (by the bisection property), and $\sqrt{L^2 - (L/2)^2} = \sqrt{3}L/2$ by the Pythagorean Theorem.

Exercise 9–3. The values are -1, 0, 0, and -1 respectively.

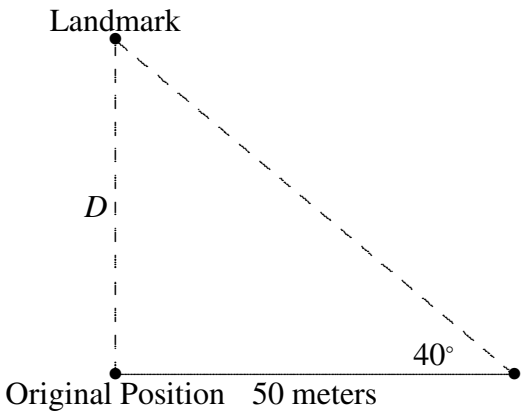
Exercise 9–4. The point (x, y) lies on the graph of the sine curve if and only if the point $(x - \pi/2, y)$ lies on the graph of the cosine curve. Since the second coordinate of these two points is the same, the desired relationship follows.

Exercise 9–5. $\tan A = \mathcal{O}/\mathcal{A}$.

§10. Applications Involving Trigonometric Functions

Some applications of trigonometric functions are given.

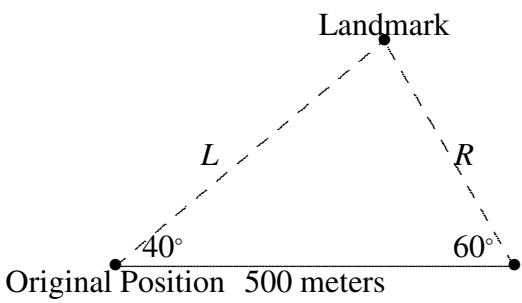
Example 10–1. In order to measure the distance D accross a river the following scheme is used. A surveyor marks a point on this side of the river directly opposite a landmark on the far side of the river. She then measures off a distance of 50 meters in a direction perpendicular to the direction that crosses the river. Finally, she uses her transit to measure the angle between this 50 meter baseline and the line of sight to the original landmark. Schematically, the situation is shown below.



Using the definition of the trigonometric functions gives $\dfrac{D}{50} = \tan 40^\circ$, so that $D = 50 \tan 40^\circ = 41.95$.

Exercise 10–1. What is the distance from the second point marked to the landmark?

Example 10–2. In a more realistic setting, the original angle would not be a right angle. With the information given below, what is the shortest distance accross the river?



In this case, drop the perpendicular from the landmark to the measured line and call its length D. Using the newly created small right triangles and the definition of the basic trigonometric functions gives $\dfrac{D}{L} = \sin 40^\circ$ and $\dfrac{D}{R} = \sin 60^\circ$. Equating the two resulting expressions for D gives $L \sin 40^\circ = R \sin 60^\circ$, or $\dfrac{L}{\sin 60^\circ} = \dfrac{R}{\sin 40^\circ}$.

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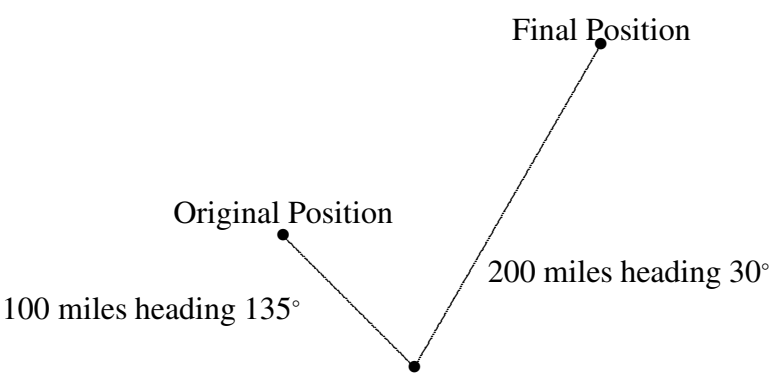
This indicates that *the ratio of the length of the side of a triangle divided by the sine of the angle opposite the side, does not depend on the side!* This fact is called the **Law of Sines**. Now the remaining angle of the original triangle is $180^\circ - 40^\circ - 60^\circ = 80^\circ$, and the length of the side opposite this angle is 500. By the Law of Sines $\dfrac{R}{\sin 40^\circ} = \dfrac{500}{\sin 80^\circ}$ so $R = \dfrac{500}{\sin 80^\circ}\sin 40^\circ = 326.35$. Similarly, $L = \dfrac{500}{\sin 80^\circ}\sin 60^\circ = 439.69$ and the distance accross the river is $D = 282.62$.

The previous example shows that the Law of Sines is useful when either 2 angles and the length of one side are known, or when the length of 2 sides and the angle opposite one of the sides is known.

Exercise 10–2. Explain in detail how the Law of Sines would be used to find the remaining angle and the lengths of the remaining 2 sides if any 2 angles and the length of one side are known.

Exercise 10–3. Explain in detail how the Law of Sines would be used to find the remaining angles and the length of the remaining side if the lengths of two sides are known, as well as the angle opposite one of these two sides.

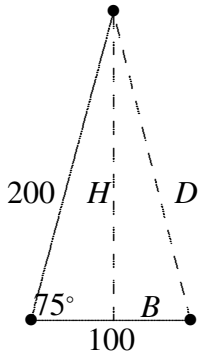
Example 10–3. In order to avoid a thunderstorm, a plane flies 100 miles on a heading of 135°, and then flies 200 miles on a heading of 30°. At the end of this time, how far is the plane from its original point? Traveling on a heading of 135° means that the plane is flying in the direction which is 135° clockwise from north (which is the direction with heading 0°). Thus the plane first flies southeast for 100 miles, and then flies slightly north of northeast for 200 miles. The picture is as follows.



Exercise 10–4. What is the angle at the base of this picture?

Now rotate the triangle and drop the perpendicular from the final position to the opposite side. The distance D is to be found. Label the length of the perpendicular

H and the length of the leg of the newly created right triangle B. The picture is below.



Examining the right triangle on the left of the picture and using the basic definitions gives $\dfrac{H}{200} = \sin 75^\circ$ and $\dfrac{100-B}{200} = \cos 75^\circ$ from which $H = 200 \sin 75^\circ$ and $B = 100 - 200 \cos 75^\circ$. Now from the right triangle on the right and the Pythagorean Theorem

$$\begin{aligned} D^2 &= H^2 + B^2 \\ &= (200 \sin 75^\circ)^2 + (100 - 200 \cos 75^\circ)^2 \\ &= (200)^2(\sin 75^\circ)^2 + (200)^2(\cos 75^\circ)^2 - 2(100)(200) \cos 75^\circ + (100)^2 \\ &= (200)^2 + (100)^2 - 2(200)(100) \cos 75^\circ. \end{aligned}$$

Computing gives $D = 199.11$. This formula is called the **Law of Cosines**: the square of the length of the side opposite an angle is equal to the sum of the squares of the lengths of the remaining sides minus twice the product of the lengths of the sides and the cosine of the angle between them.

Exercise 10–5. If there were no thunderstorm, what heading would the pilot fly from the original point to reach the final point?

The Law of Cosines can be used effectively when two sides of a triangle and the included angle are known, and also when all three sides of a triangle are known.

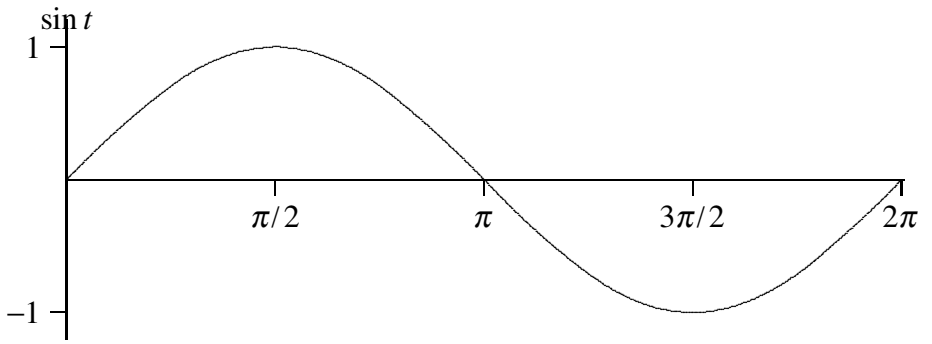
Exercise 10–6. How would the Law of Cosines be used if all three sides of the triangle are known?

The trigonometric functions sine and cosine are often used as models of periodic phenomena. As was seen earlier, the cosine function can be obtained by sliding the sine function to the right. For this reason, attention can be focused on the sine function alone.

One of the basic situations in which the sine function is used is in the modeling of sound waves. Any device which produces sound does so by causing the air to vibrate. These vibrations are transmitted to the ear and cause a corresponding vibration of the eardrum. These vibrations are interpreted by the brain as sound.

There are two basic features of sound that can be perceived: the loudness of the sound and the pitch of the sound. The loudness of the sound corresponds to the magnitude of the vibration: sound waves with larger vibrations are perceived as louder sounds. The pitch of the sound is determined by the number of vibrations that impinge upon the ear in a given length of time: the greater the number the higher the pitch.

The problem of translating these physical characteristics into a mathematical model will now be considered. In the present context the independent variable will be denoted by t and be thought of as representing time. For reference purposes, the basic graph is shown below.



From the graph, the magnitude of the vibration of this basic sine function is the difference between its maximum value 1 and its minimum value -1. The **amplitude** of the vibration is defined to be half of this peak to trough diistance. Hence the amplitude of the basic sine function is 1. The graph of the basic sine function repeats itself every 2π time units. The **period** of the basic sine function is therefore 2π. The **frequency** of oscillation is the number of complete cycles completed in one unit of time. For the basic sine function, one complete cycle requires 2π time units; thus the frequency of the basic sine function is $1/2\pi$ cycles per unit time. In connection with the earlier discussion of sound, the amplitude of the sine curve is related to the volume of the sound while the frequency of the sine curve is related to the pitch of the sound.

Example 10–4. The function $3\sin(5t)$ has an amplitude of 3 and a frequency of $5/2\pi$ cycles per unit time.

Example 10–5. The pitch change in music called an octave corresponds to the doubling of the frequency of sound. Each octave is divided into 12 equal intervals of pitch change which together make up the one octave chromatic scale. The ratio of sucessive frequencies in the chromatic scale is therefore $2^{1/12}$. The note called middle C corresponds to a frequency of about 262 cycles per second. (The basic unit of frequency is the hertz, which is one cycle per second.)

Exercise 10–7. What sine function has a frequency corresponding to middle C?

Exercise 10–8. The C major chord consists of the notes C, E, and G. What function would sound a C major chord?

As was seen earlier, the cosine curve can be obtained by sliding the sine curve to the left along the axis. In this context, sliding horizontally is called a **phase shift**.

Exercise 10–9. If the graph of the curve $\sin t$ is slid $\pi/2$ units to the left, what is the function which has this new curve as its graph?

Exercise 10–10. If the graph of the curve $2\sin(5t)$ is slid $\pi/2$ units to the left, what is the function which has this new curve as its graph?

To develop some additional properties of the trigonometric functions and their graphs, the connection between the trigonometric functions and exponential function will be developed. This will require an excursion into the study of the complex numbers.

Problems

Problem 10–1. A rocket is fired from level ground and rises along a line making an angle of 75° with the ground. After travelling 5,000 feet, how high is the rocket above the ground?

Problem 10–2. A drawbridge consists of 2 pivoting sections, each of which is 75 feet long. Each section can be rotated upwards through an angle of 45° with the horizontal. If the water level is 15 feet below the level of the bridge, how high above the water is the tip of the pivoting section when the bridge is fully open? What is the distance between the two ends of the pivoting sections when the bridge is fully open? Could a barge carrying a rectangular cargo that is 30 feet above the waterline and 40 feet wide pass under the open bridge?

Problem 10–3. From a space station 380 miles above the surface of the earth, the angle between the line from the station to the point beneath it on the earths surface and the line from the station to the earths horizon is 65.8°. From this information, what is the approximate radius of the earth?

Problem 10–4. A motorist is traveling directly toward a mountain on a flat highway at a rate of 60 miles per hour. Initially, the motorist must look up at an angle of 20° to view the top of the mountain; 5 minutes later she must look up at an angle of 40° to view the top. How high is the mountain above the road?

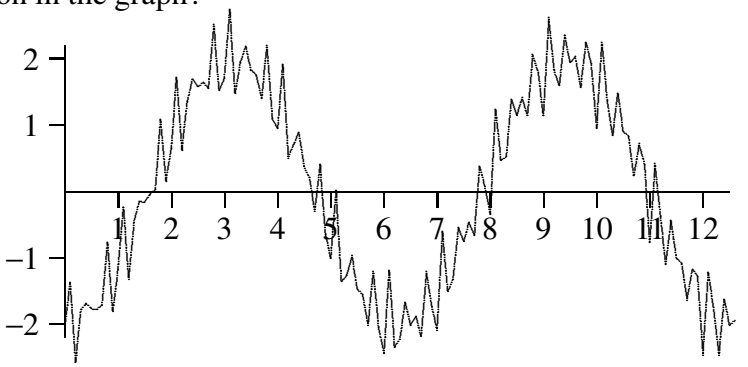
Problem 10–5. On a roof that makes a 25° angle with the horizontal, a solar panel is to be mounted so that the panel makes an angle of 45° with the horizontal. How long should a vertically placed prop be in order to mount the 10 foot wide panel?

Problem 10–6. Two observers 1 mile apart on level ground see a hot air balloon. One observer must look up through an angle of 37° to see the balloon, while the other must look up through an angle of 53° to see the balloon. How high is the balloon above the ground? What assumptions have you made?

Problem 10–7. An airplane at an altitude of 10,000 feet spots two ships on the ocean below. To see the first ship, the observer must look down through an angle of 39°, while to see the second the observer must look down through an angle of 19°. To fly from its present position to the first ship, the airplane would have to take a heading of 220°. To fly from its present position to the second ship, the airplane would have to take a heading of 80°. How far apart are the ships?

Problem 10–8. In the graph below, what is the approximate amplitude and period

of the function in the graph?



Problem 10–9. Suppose a single sine curve of the form $A\sin(Bt+C)$ is to be fitted to the graph in the previous problem. What would be the approximate values of A, B, and C?

Solutions to Problems

Problem 10–1. By drawing a right triangle the height H above the ground satisfies $H/5000 = \sin 75°$.

Problem 10–2. The height of the tip of a section above the water is $15 + 75 \sin 45°$. For the other questions, choose a convenient coordinate system and find the coordinates of selected points.

Problem 10–3. There is a right triangle here with one side of lenght R (the earth radius) and the hypothenuse of length $R + 380$.

Problem 10–4. There are 2 right triangles here, with the height of the mountain as a common leg. The height of the mountain is about 16969 feet.

Problem 10–5. Use the Law of Sines.

Problem 10–6. The balloon should be in the same vertical plane as the two observers, and the observations should be made at the same point in time. Under these conditions the Law of Sines can be used.

Problem 10–7. The airplane and the ships are not in the same vertical plane. So first find the distance from each ship to the spot immediately below the aircraft. These distances and the heading information can be used with the Law of Cosines.

Problem 10–8. The amplitude is approximately 2 and the period is about 6.

Problem 10–9. $A = 2$ and $B = 1$, from the previous answer. The phase shift $C = -1.5$.

Solutions to Exercises

Exercise 10–1. If H is the hypotenuse of the right triangle of the picture, then $\frac{50}{H} = \cos 40^\circ$, so that $H = 65.27$.

Exercise 10–2. The third angle can first be found using the fact that the sum of the measures of the angles in a triangle is π radians. The ratio of the length of the known side to the sine of the angle opposite it is now known, and the Law of Sines permits the computation of the length of the remaining sides using the sine of the remaining angles.

Exercise 10–3. The angle opposite the second known side can first be found using the Law of Sines. Now use the method of the previous exercise.

Exercise 10–4. Drawing the compass directions at the base vertex shows that the incoming line from the northwest makes an angle of 45° with the north-south line, while the departing line to the northeast makes an angle of 30° with the north-south line. Hence the angle at the base vertex is 75°.

Exercise 10–5. Using the distance D and the Law of Sines shows that the angle in the lower right of the second picture is 76° (approximately), so the heading would have been $135^\circ - 76^\circ = 59^\circ$.

Exercise 10–6. The Law of Cosines can be used to find the cosine of the angle between any two sides. Once one angle is known, the others can be found by the Law of Sines. To avoid ambiguity when using the Law of Sines, it is best to find the largest angle using the Law of Cosines. The largest angle is the angle opposite the longest side.

Exercise 10–7. The sine function $\sin(2\pi 262t)$.

Exercise 10–8. The function $\sin(2\pi \cdot 262 \cdot t) + \sin(2\pi \cdot 262 \cdot 2^{4/12}t) + \sin(2\pi \cdot 262 \cdot 2^{7/12}t)$, since E and G are the fourth and seventh notes in a chromatic scale beginning at C.

Exercise 10–9. The point (x, y) is on the new curve if and only if the point $(x + \pi/2, y)$ is on the graph of $\sin t$. Hence the function required is $\sin(x + \pi/2)$.

Exercise 10–10. The function required is $2\sin(5(t + \pi/2)) = 2\sin(5t + 5\pi/2)$.

§11. Euler's Identity and the Exponential and Trigonometric Functions

Further properties of the trigonometric functions can most easily be developed by making use of the connection between the trigonometric functions and the exponential function first given by Leonhard Euler about 300 years ago. This connection makes use of some basic properties of complex numbers.

There is no real number whose square is -1. This deficiency can be remedied by creating a new number system, called the **complex numbers** and denoted by $\mathbf{C}$, which consists of all numbers of the form $a+bi$ where a and b are real numbers and i is a new symbol with the property that $i^2 = -1$. In the complex number $a+bi$, the real number a is called the **real part** and the real number b is called the **imaginary part**.

Exercise 11–1. What are the real and imaginary parts of $3+4i$?

Exercise 11–2. Write $\mathbf{C}$ as a set.

Basic arithmetic with complex numbers is done like arithmetic with polynomials, treating i as a symbol with the property that $i^2 = -1$. Thus $(2+3i)+(4-5i) = 6-2i$ and $(2+3i)(4-5i) = 8-10i+12i-15i^2 = 23-3i$. There is no sense of ordering of complex numbers as there is with real numbers; a statement such as $2+3i < 4-6i$ has no meaning. As partial compensation, the **modulus** or **absolute value** of a complex number is defined by the formula $|a+bi| = \sqrt{a^2+b^2}$.

Exercise 11–3. What is $(2-3i)(4-2i)$? What is $|3-5i|$?

As was the case with the real numbers, the set of complex numbers also has a visual representation. The visual representation $\mathbf{C}$ is as a two dimensional plane: the complex number $a+bi$ is represented as the point $(a,b) \in \mathbf{R}^2$.

Exercise 11–4. What is the geometric interpretation of $|a+bi|$?

Exercise 11–5. What geometric object is the set of all complex numbers of modulus 1?

The last exercise shows that any complex number of modulus 1 can be written in the form $\cos t + i \sin t$ for some real number t. Thus any complex number $a+bi$ can be written in the form $A(\cos t + i \sin t)$ for some real number $A \geq 0$ and some real number t.

Exercise 11–6. How are A and t determined from a and b?

The remarkable connection discovered by Euler is that for any real number t, $e^{it} = \cos t + i \sin t$. Using this fact and the familiar properties of the exponen-

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tial function allows a deeper understanding of the behavior of the trigonometric functions.

Exercise 11–7. True or False: $2\cos t = e^{it} + e^{-it}$.

The basic methodology for using Euler's identity is this. Two complex numbers are equal if and only if the two numbers have the same real part and the same imaginary part. So to study properties of cosine, simply use the fact that $\cos t$ is the real part of e^{it}. After doing manipulations with the complex exponential, look at the real part of the result. This must be equal to $\cos t$.

Example 11–1. How can $\cos(A+B)$ be expressed in terms of trigonometric functions of A and B separately? First note that $\cos(A+B)$ is the real part of $e^{i(A+B)}$, by Euler's identity. Now use properties of exponential and Euler's identity again to obtain

$$\begin{aligned} e^{i(A+B)} &= e^{iA}e^{iB} \\ &= (\cos A + i\sin A)(\cos B + i\sin B) \\ &= (\cos A\cos B - \sin A\sin B) + i(\cos A\sin B + \cos B\sin A). \end{aligned}$$

The real part of this last expression must be equal to $\cos(A+B)$. Hence $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

Exercise 11–8. What is an expression for $\sin(A+B)$?

Similar manipulations can be used to simplify sums of trigonometric functions.

Example 11–2. In the study of music, dissonance occurs when two notes of nearly equal pitch are sounded simultaneously. Suppose the two notes have frequencies of 440 and 441 hertz. The two note chord would then be $\sin(2\pi 440t) + \sin(2\pi 441t)$, which is the imaginary part of

$$\begin{aligned} e^{i880\pi t} + e^{i882\pi t} &= e^{i881\pi t}(e^{-i\pi t} + e^{i\pi t}) \\ &= 2\cos(\pi t)e^{i881\pi t} \\ &= 2\cos(\pi t)\cos(881\pi t) + i2\cos(\pi t)\sin(881\pi t). \end{aligned}$$

Hence $\sin(2\pi 440t) + \sin(2\pi 441t) = 2\cos(\pi t)\sin(881\pi t)$. This is a sine wave with time dependent amplitude. The amplitude has a frequency of $1/2$ hertz. This phenomenon is heard as beats.

Problems

Problem 11–1. If c is a complex number, how are $|c|$ and $|5c|$ related?

Problem 11–2. The function $5\cos t$ is the real part of what complex exponential?

Problem 11–3. Write $\cos t + \cos 2t$ as a product of trigonometric functions.

Problem 11–4. It is also possible to directly use the facts that $\cos t = (e^{it} + e^{-it})/2$ and $\sin t = (e^{it} - e^{-it})/2$ to manipulate the trigonometric functions. Use this dictionary relationship between the exponential and trigonometric functions to write the product $\sin A \cos B$ as a sum of trigonometric functions.

Problem 11–5. Electricity can be readily transmitted in the form of alternating current. The voltage in an alternating current circuit is a sine wave. The amplitude of the sine wave is the peak-to-peak voltage, which is commonly 170 volts in the United States. (Since the voltage in an alternating current circuit is not constant, the usual voltage specification is to use the root mean square voltage. This is the peak-to-peak voltage divided by $\sqrt{2}$. The root mean square voltage is $170/\sqrt{2} = 120$ approximately.) The frequency of the sine wave is 60 hertz. If time is measured in seconds, write an expression for the voltage as a function of time.

Problem 11–6. Some household appliances require 120 volts (root mean square) while others require 220 volts. In order to avoid using wires with two different voltages the following scheme is used. Two 120 volt wires are used, but the voltage in one wire has a phase difference of $\pi/3$ times the frequency from that in the other. When the two wires are used together, the voltages add. Write expressions for the voltages in each wire, and an expression for the sum of the two voltages. What is the peak-to-peak voltage of this combination?

Solutions to Problems

Problem 11–1. $|5c| = 5|c|$.

Problem 11–2. $5\cos t$ is the real part of $5e^{it}$.

Problem 11–3. Write each piece as the real part of an exponential, then factor and combine. This gives $\cos t + \cos 2t = 2\cos(t/2)\cos(3t/2)$.

Problem 11–4. Write $\sin A = (e^{iA} - e^{-iA})/2i$ and $\cos B = (e^{iB} + e^{-iB})/2$, multiply and collect terms to obtain $\sin A \cos B = (\sin(A+B) + \sin(A-B))/2$.

Problem 11–5. The voltage would be $170\sin(120\pi t)$.

Problem 11–6. The expressions are $170\sin(120\pi t)$, $170\sin(120\pi(t - \pi/3))$, and $170\sin(120\pi t) + 170\sin(120\pi(t - \pi/3))$. The peak to peak voltage is $340\cos(120\pi \cdot \pi/6)$ volts.

Solutions to Exercises

Exercise 11–1. The real part is 3 and the imaginary part is 4.

Exercise 11–2. $\mathbf{C} = \{a + bi : a \in \mathbf{R} \text{ and } b \in \mathbf{R}\}$.

Exercise 11–3. $(2 - 3i)(4 - 2i) = 2 - 18i$ and $|3 - 5i| = \sqrt{34}$.

Exercise 11–4. The geometric interpretation of $|a + bi|$ is the distance from (a, b) to the origin $(0, 0)$.

Exercise 11–5. This is the set $\{(a, b) : \sqrt{a^2 + b^2} = 1\}$, which is the unit circle.

Exercise 11–6. The number A is uniquely determined by $A = |a+bi| = \sqrt{a^2 + b^2}$ and there are many choices for t, the conditions which must be satisfied are that $\cos t = a/\sqrt{a^2 + b^2}$ and $\sin t = b/\sqrt{a^2 + b^2}$.

Exercise 11–7. True, using Euler's identity twice.

Exercise 11–8. Look at the imaginary parts to obtain $\sin(A + B) = \cos A \sin B + \cos B \sin A$.

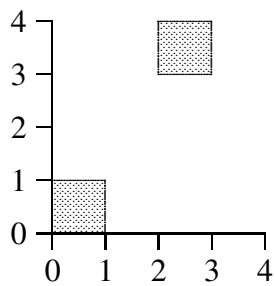
§12. The Geometry of Complex Arithmetic

As a prelude to studying functions in higher dimensional spaces some geometric properties of the arithmetic of complex numbers is discussed.

Since a complex number can be visualized as a point in the plane, the basic arithmetic operations on complex numbers can also be visualized as operations on points in the plane. This allows the arithmetic operations to have a geometric interpretation.

Example 12–1. What geometric interpretation can be given to adding $2+3i$ to other complex numbers? Given a particular number $a + bi$, the sum $(a + bi) + (2 + 3i) = (2+a)+i(3+b)$ is easily found. In order to understand this operation geometrically, this operation will be applied to a collection of complex numbers. To do this simply, suppose the numbers $a + bi$ are all chosen from the set of complex numbers which are represented visually as the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ in the plane. The picture is shown below.

Addition of $2 + 3i$ Point by Point



The addition of $2 + 3i$ to each point in the initial square has the effect of sliding the square right 2 units and up 3 units.

Exercise 12–1. Check a few points of your own choosing and see that this is the case.

The process that produced the picture of the example should be recognized as the operation of a function. In this case, the rule of the function is $a+bi \mapsto (a+2)+(b+3)i$.

Exercise 12–2. What is the domain of this function? What is the range of this function?

This function would be more conventionally written as $S(x+iy) = (2+x)+i(3+y)$. In view of the geometric interpretation, a function could also be defined with domain $\mathbf{R}^2$ and range $\mathbf{R}^2$ and rule $S(x, y) = (x+2, y+2)$. This would directly give the sliding function in its natural visual setting.

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Exercise 12–3. Are the two functions called S here the same?

Example 12–2. If a real number is fixed, say 5, and complex numbers are multiplied by 5, what is the geometric interpretation? Here the function is defined by $M(x+iy) = 5x + 5yi$. The original square in the picture above is then stretched by a factor of 5 in all directions.

Exercise 12–4. What happens if the real number selected is between 0 and 1? Between -1 and 0?

Exercise 12–5. What function with domain and range $\mathbf{R}^2$ would be defined by this multiplication?

Example 12–3. Finally, suppose a fixed complex number $e^{i\pi/4}$ is used as the multiplier rather than a real number, as in the last example. Since any complex number can be written in the form $Ae^{i\theta}$ for some real number A and some real number θ, multiplication of this complex number by $e^{i\pi/4}$ gives $Ae^{i(\theta+\pi/4)}$. This rotates the original square through an angle of 45° in the counter-clockwise direction with the origin as the pivot point.

Exercise 12–6. Draw the picture of the original unit square after the multiplication has been done.

Exercise 12–7. What is the formula for the function with domain and range $\mathbf{R}^2$ which accomplishes this same rotation?

The basic arithmetic operations on complex numbers have now been given visual interpretations. In doing so, an interesting collection of functions has been developed. These functions can be viewed as either having domain and range $\mathbf{C}$, or as having domain and range $\mathbf{R}^2$. This second point of view will be analyzed a bit further in the next section.

Problems

Problem 12–1. The complex number $a - bi$ is called the **conjugate** of the number $a + bi$ and is denoted by $\overline{a + bi}$. What happens to the original square if each point in it is mapped into its conjugate? Write a formula for this function as a function with domain $\mathbf{R}^2$ and range $\mathbf{R}^2$.

Problem 12–2. True or False: For any complex number $a + ib$, $|a + ib|^2 = (a + ib)(\overline{a + ib})$.

Problem 12–3. Explain how the original square could be rotated through an angle of 30° with the origin as a pivot, and write the function to accomplish this both as a function with domain and range $\mathbf{C}$ and as a function with domain and range $\mathbf{R}^2$.

Problem 12–4. Explain how the original square could be rotated through an angle of 30° with the point $(1, 1)$ as the pivot, and write the function to accomplish this in both real and complex form.

Solutions to Problems

Problem 12–1. The square is reflected through the x axis. In real form the function is $R(x, y) = (x, -y)$.

Problem 12–2. True.

Problem 12–3. In complex form the function is $F(x + iy) = e^{i\pi/6}(x + iy)$.

Problem 12–4. First slide the square so that the point originally at $(1, 1)$ is at the origin, then rotate, then slide back.

Solutions to Exercises

Exercise 12–2. The domain and range are both **C**.

Exercise 12–3. No, because neither the domain, range, or rule is the same. Even so, the behavior of the functions is visually the same.

Exercise 12–4. If the real number is between 0 and 1, the square shrinks by that factor in all directions, while if the number is between -1 and 0, there is both shrinking and reflection through the origin.

Exercise 12–5. $M(x, y) = (5x, 5y)$.

Exercise 12–7. $R(x, y) = (x\cos(\pi/4) + y\sin(\pi/4), y\cos(\pi/4) - x\sin(\pi/4))$.

§13. Geometry in Higher Dimensions

Most of the functions studied so far have had one dimensional domains and one dimensional ranges. As was seen in the previous section, functions have many applications in situations in which either the domain or range or both are higher dimensional. The discussion of properties of such functions depends on the geometry of higher dimensional space itself. The basic aspects of this geometry will be discussed here.

The familiar two dimensional plane $\mathbf{R}^2$ is the set of all ordered pairs of real numbers. Three dimensional space,$\mathbf{R}^3$, is the set of all ordered triples of real numbers. In set notation $\mathbf{R}^3 = \{(x, y, z) : x \in \mathbf{R} \text{ and } y \in \mathbf{R} \text{ and } z \in \mathbf{R}\}$. Generally, the d dimensional space $\mathbf{R}^d$ is the set of ordered d tuples of real numbers. In set notation $\mathbf{R}^d = \{(x_1, \ldots, x_d) : x_1 \in \mathbf{R} \text{ and } x_2 \in \mathbf{R} \ldots \text{and } x_d \in \mathbf{R}\}$. The elements of the space $\mathbf{R}^d$ are called **points** or **vectors**.

Example 13–1. The point $(1, 2, 3, 4)$ is a vector in $\mathbf{R}^4$, while $(-3, 7, 10)$ is a point in $\mathbf{R}^3$.

Example 13–2. The point $(2, 3) \in \mathbf{R}^2$ can have two different interpretations. The first interpretation is that $(2, 3)$ is simply a point in the plane. The second interpretation comes about in the following way. Imagine standing at the origin and facing toward the point $(2, 3)$. The point $(2, 3)$ determines a direction from the origin, and also a distance (or magnitude) in that direction. In this way the point $(2, 3)$ specifies both a magnitude and a direction. The same two interpretations apply in spaces of any dimension.

In order to capture the idea of magnitude, the **norm** or length of a vector is used. In the two dimensional case $\| (2, 3) \| = \sqrt{2^2 + 3^2}$, and generally, for a vector $v = (v_1, \ldots, v_d) \in \mathbf{R}^d$, $\| v \| = \sqrt{v_1^2 + \ldots + v_d^2}$.

Exercise 13–1. What is the geometric interpretation of $\| v \|$?

Example 13–3. The interpretation of a vector as representing both a magnitude and a direction is often used in applications. Physical quantities such as forces and velocities are represented by vectors because these physical quantities represent both a magnitude and a direction.

The connection between complex numbers and their visual representation as points in the plane suggests that certain operations on vectors can be valuable. In particular, the addition formula for complex numbers $(2 + 3i) + (3 - 5i) = 5 - 2i$ suggests the corresponding operation on points $(2, 3)+(3, -5) = (5, -2)$. This method of addition readily generalizes to a space of any dimension. The corresponding

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notion of subtraction is also apparent. Multiplication of a complex number by a real number, as in $5(2 + 3i) = 10 + 15i$, suggests the corresponding operation on points: $5(2, 3) = (10, 15)$. This method of **scalar multiplication** also works in any dimension.

Exercise 13–2. What is $(1, 2, 3, 4) + (2, -3, 5, 7)$? What is $3(1, 0, -3, 5)$?

Exercise 13–3. What geometric object is the set of all real multiples of the vector $(2, 3)$?

Example 13–4. One extension of the idea of the last exercise is a re-examination of the interpretation of a vector as representing both a magnitude and a direction. Any non-zero vector v can be written $v = \| v \| (v/\| v \|)$. Clearly $\| v \|$ is the magnitude of v. The vector $v/\| v \|$ is a vector in the same direction as v which has magnitude 1. This means that $v/\| v \|$ can be thought of as the direction associated with the vector v. This way of viewing a vector is often useful in finding the vector associated with a physical quantity.

Example 13–5. A child pulls on a wagon handle with a force of 30 pounds. The handle makes an angle of 40° with the horizontal. What vector represents the force exerted by the child on the wagon itself? The magnitude of the force is 30 pounds. The direction in which the force acts is in a direction making an angle of 40° with the horizontal; this direction is represented by the vector $(\cos 40^\circ, \sin 40^\circ)$. (Notice that this is a vector of magnitude 1.) So the force exerted by the child on the wagon is $30(\cos 40^\circ, \sin 40^\circ) = (22.98, 19.28)$.

Exercise 13–4. What vector represents the direction of a force acting at an angle of θ with the horizontal?

Vectors of length one are called **unit vectors**. Unit vectors are typically used to indicate a direction.

Example 13–6. In higher dimensional spaces the specification of simple geometric objects becomes more difficult. How should a line be specified in 3 dimensional space? Certainly a single relationship between the 3 coordinates of a point on the line is insufficient. The line through the origin and the point $(2, 3)$ is the set of all multiples of the vector $(2, 3)$. This set can be written $\{t(2, 3) : t \in \mathbf{R}\}$. This is called a **parametric representation** of the line. The variable t in this representation is the parameter, which is allowed to vary freely over the set of all real numbers. Since this notation is rather bulky, usually just the formula part of the set is specified with the remainder being understood. The **parametric equation** of the line through the origin and $(2, 3)$ is $t(2, 3)$. (The use of the term equation is somewhat misleading, since there is in fact no equation!)

Exercise 13–5. What is the parametric equation of the line through the origin and the point $(2,-1)$?

Exercise 13–6. Is $s(4,-2)$ the parametric equation of the line through the origin and the point $(2,-1)$?

Example 13–7. In the earlier discussion of complex arithmetic, addition had the geometric interpretation of sliding. The same is true in spaces of any dimension. This idea can be used to find the parametric equation of the line through any two points. To find the parametric equation of the line through the points $(1,3)$ and $(2,5)$, reason as follows. First slide the two points by the same amount so that one of them is at the origin. Sliding $(1,3)$ to the origin gives the other point as $(1,2)$ after sliding. The line through the origin and $(1,2)$ is given parametrically by $t(1,2)$. Now slide this line back to obtain the parametric equation of the original line: $(1,3)+t(1,2)$.

Exercise 13–7. Repeat this argument sliding $(2,5)$ to the origin instead. What parametric equation is obtained?

Generally, if u and v are vectors, the parametric equation of the line through u and v is $u+t(v-u)$. This line can also be appropriately called the line through u in the direction $v-u$. This is because $v-u$ is the direction in which the line is stretching.

Exercise 13–8. What is the parametric equation of the line through $(1,2,3,4)$ and $(5,6,7,8)$?

Exercise 13–9. What is the parametric equation of the line through $(1,2,3,4)$ in the direction $(1,0,0,-1)$?

Another way of looking at the parametric equation of a line is as a function. Suppose the function L is defined by the rule $L(t)=(1,2)+t(2,3)$. Then L has domain $\mathbf{R}$ and the range of L is the line through $(1,2)$ in the direction $(2,3)$ inside of $\mathbf{R}^2$. This coincides with intuition which says that lines are the images of one dimensional objects.

Example 13–8. A two dimensional plane is determined by 3 points. These three points determine two directions in which the plane stretches. With this idea in mind the parametric equation of a plane can be found. For concreteness, suppose the 3 points are $(1,2,3)$, $(4,5,6)$, and $(3,2,1)$. Choose the first point arbitrarily as the base point. From this base point the two directions are $(4,5,6)-(1,2,3)=(3,3,3)$ and $(3,2,1)-(1,2,3)=(2,0,-2)$. The parametric equation of the plane is then $(1,2,3)+t(3,3,3)+s(2,0,-2)$.

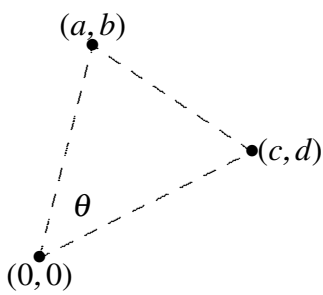
Exercise 13–10. Write this plane as a set.

Exercise 13–11. Find two other parametric equations for the same plane.

Exercise 13–12. Define a function with domain $\mathbf{R}^2$ and this plane as its range.

The basic geometry of higher dimensional spaces has now been examined and has been seen to be much the same as in the familiar 2 and 3 dimensional cases. The important idea of measuring angles has been absent so far. The next example illustrates how this idea can be developed.

Example 13–9. In order to understand how the idea of angle can be developed in higher dimensional spaces, the measurement of angles in two dimensional space is re-examined. What is meant by the angle between two vectors? A vector determines a direction by using the origin as one reference point and the point specified by the vector as the other reference point; the direction is then determined by these two points. The angle between two vectors is the angle formed by the two points specified by the vectors with the origin as the vertex of the angle. Suppose that two vector (a, b) and (c, d) are given, as in the following picture.



The angle θ between these two vectors can be found using the Law of Cosines, since the lengths of all three sides of the triangle are known. Hence

$$\| (a,b) - (c,d) \|^2 = \| (a,b) \|^2 + \| (c,d) \|^2 - 2\| (a,b) \| \| (c,d) \| \cos\theta.$$

On the other hand, $\| (a,b) - (c,d) \|^2$ can also be computed directly to give

$$\begin{aligned} \| (a,b) - (c,d) \|^2 &= \| (a-c, b-d) \|^2 \\ &= (a-c)^2 + (b-d)^2 \\ &= a^2 - 2ac + c^2 + b^2 - 2bd + d^2 \\ &= \| (a,b) \|^2 + \| (c,d) \|^2 - 2(ac + bd). \end{aligned}$$

Comparing these two expressions for the same quantity gives

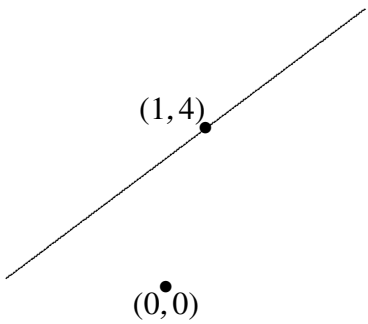
$$\cos\theta = \frac{ac + bd}{\| (a,b) \| \| (c,d) \|}$$

from which the angle θ can be found. The important point is to notice that the angle θ depends only on the two norms and the quantity $ac + bd$.

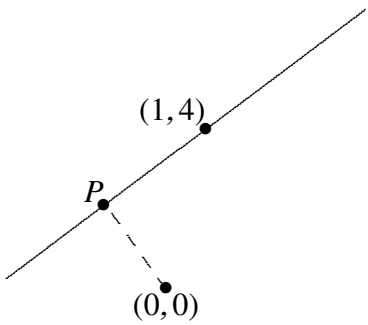
The previous example suggests that the quantity $ac+bd$ carries the trigonometric information in this setting. Define the **dot product** of the two vectors by the formula $(a, b)\bullet(c, d) = ac+bd$. Notice that the dot product of two vectors is a number obtained by multiplying the corresponding entries in the two vectors and then adding. The dot product is defined similarly in spaces of any dimension. The important geometric fact is that *two vectors are perpendicular if and only if the dot product of the two vectors is zero*. This is very easy to check computationally.

Exercise 13–13. What is $(1, 2)\bullet(3, 4)$? What is $(1, 2, 3, 4)\bullet(-1, 2, -2, 5)$? Are the vectors $(1, 2, 3)$ and $(-3, 2, -1)$ perpendicular?

Example 13–10. Many important applied problems can be reduced to the following simple geometric framework. What is the point on the line through $(1, 4)$ in the direction $(4, 3)$ which is closest to the origin? The initial picture is as shown below.



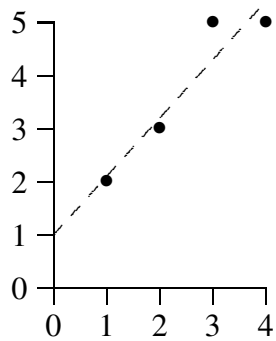
The point P which is to be found has two properties: P must lie on the line, and the line connecting P to the origin must form a right angle with the given line. The picture with the point P included is as below.



These two geometric requirements translate into the two algebraic requirements $P = (1, 4) + m(4, 3)$ for some number m, and $P\bullet(4, 3) = 0$. Substituting the formula for P into the second requirement and computing gives $m = -16/25$, and thus $P = (1, 4) - (16/25)(4, 3) = (-39/25, 52/25)$.

Example 13–11. In the experimental sciences, functions are often used to summarize data. Suppose an experimenter has observed the following pairs of observations on the variables x and y: $(1,2)$, $(2,3)$, $(3,5)$, $(4,5)$. A simple plot is made of these data points, and is shown below.

Experimental Data



It seems reasonable to assume that there is a linear relationship between x and y (perhaps as given by the dashed line), and that the deviation from that line is due to experimental error. Based on this data, what is the best estimate of that linear relationship? The term 'best' is subject to interpretation. One conventional interpretation is that the best line is given by the **least squares criterion**: the best fitting curve to the data is the one that minimizes the sum of the squares of the vertical distances from the data to the curve. Suppose the curve $y = ax + b$ is to be fitted to the given data using the least squares criterion. The numbers a and b will then be chosen so that

$$(a \cdot 1 + b - 2)^2 + (a \cdot 2 + b - 3)^2 + (a \cdot 3 + b - 5)^2 + (a \cdot 4 + b - 5)^2$$

is as small as possible. To find a and b, this minimization problem is given a geometric interpretation. The quantity to be minimized is the distance from the point $a(1,2,3,4) + b(1,1,1,1)$ to the point $(2,3,5,5)$. The required values of a and b therefore determine the point in the plane through $(0,0,0,0)$, $(1,2,3,4)$ and $(1,1,1,1)$ which is closest to the point $(2,3,5,5)$. This is exactly the problem of the previous example.

Exercise 13–14. Use the geometric picture to find a and b.

The dot product also plays an important role in decomposing a vector representing a physical quantity into components acting in specified directions.

Example 13–12. The frictional force between a block and an incline is determined by the force acting perpendicular to the incline. This is the force that is acting to push the block and incline into contact. Suppose a block weighing 100 pounds rests on a 30° incline and is acted on only by gravity. Since gravity acts as a force

toward the center of the earth, the vector representing the gravitational force on the block is $(0, -100)$. The incline is in the direction $(\cos 30^\circ, \sin 30^\circ)$ and the direction perpendicular to the incline is $(\cos 120^\circ, \sin 120^\circ)$. The objective is to express the gravitational force in terms of these two vectors, that is, to write

$$(0, -100) = a(\cos 30^\circ, \sin 30^\circ) + b(\cos 120^\circ, \sin 120^\circ)$$

for some numbers a and b. To find a and b conveniently, compute the dot product of both sides of this equation first with the vector $(\cos 30^\circ, \sin 30^\circ)$ and then with the vector $(\cos 120^\circ, \sin 120^\circ)$. Since these two vectors have magnitude 1 and are perpendicular this gives $a = -100 \sin 30^\circ$ and $b = -100 \sin 120^\circ$. The force pushing the block in contact with the incline is therefore $-100 \sin 120^\circ(\cos 120^\circ, \sin 120^\circ) = (43.30, -75)$, which has magnitude 86.60.

Exercise 13–15. What is the force along the incline?

Problems

Problem 13–1. True or False: The two vectors $(1,2)$ and $(3,5)$ are parallel.

Problem 13–2. True or False: The two vectors u and v are parallel if and only if u is a multiple of v.

Problem 13–3. Is there a single line which passes through the points $(1,2,3)$, $(2,3,5)$, and $(2,0,4)$?

Problem 13–4. Find two different non-zero vectors that are perpendicular to the vector $(1,3,-2)$.

Problem 13–5. Find a single non-zero vector which is perpendicular to both of the vectors $(1,2,0)$ and $(3,-2,4)$.

Problem 13–6. A quarterback releases a football with a speed of 40 feet per second at an angle of 50° with the ground. What vector would be used to represent the velocity of the football?

Problem 13–7. A jet airplane approaches a landing strip at a speed of 160 miles per hour and at an angle of 7.5° with the horizontal. What vector could be used to represent the velocity of the jet?

Problem 13–8. Two tugboats are towing a large ship into port. One tug exerts a force of 4000 pounds at an angle of 30° to the right of the bow of the large ship. The smaller tug is only able to exert a force of 3200 pounds on its towing cable. What angle should this towing cable for the small tug make to the left of the bow of the large ship in order to keep the ship moving in a straight line?

Problem 13–9. In order to simulate the reduced level of gravity on the moon, an incline is used. If the incline makes an angle of θ° with the horizontal and the astronaut with equipment weighs 250 pounds, what is the magnitude of the force between the astronauts shoes and the incline? What angle should be used in order to make this force equal to one-sixth of his earth weight?

Problem 13–10. An airplane is flying with an air speed of 250 miles per hour with a heading of 50°. A 40 mile per hour wind is blowing directly from the west. What is the true course and ground speed of the plane?

Problem 13–11. Find the line with equation of the form $y = mx$ which best fits the data $(1,3)$, $(-3,-4)$, $(2,2)$, $(4,5)$ in the sense of least squares.

Problem 13–12. Find the parabola with equation of the form $y = ax^2 + bx + c$ which best fits the data of the previous problem in the sense of least squares.

Problem 13–13. The function $R(x, y, z)$ rotates the point (x, y, z) through an angle of $\pi/6$ with the z axis as the pivot. Find a formula for the function R.

Solutions to Problems

Problem 13–1. False. Parallel vectors must point in the same (or opposite) direction. This means that two vectors are parallel if and only if the corresponding points lie on the same line through the origin.

Problem 13–2. True. The two points u and v are on the same line through the origin if and only if one of them is a multiple of the other.

Problem 13–3. If there is such a line, the directions from any one of the points to the other two points must be parallel. Since $(2,3,5)-(1,2,3) = (1,1,2)$ and $(2,0,4)-(1,2,3) = (1,-2,1)$ and two directions are not parallel, then there is no such line.

Problem 13–4. The vector (a,b,c) is perpendicular to $(1,3,-2)$ if and only if $(a,b,c)\bullet(1,3,-2) = 0$, and this occurs if and only if $a+3b-2c = 0$. There are several choices of a, b, and c for which this equation holds.

Problem 13–5. There are two equations the coordinates of the desired vector must satisfy.

Problem 13–6. The velocity is $40(\cos 50^\circ, \sin 50^\circ)$.

Problem 13–7. The velocity is $160(\cos(-7.5^\circ), \sin(-7.5^\circ))$. The negative sign arises since the jet is landing.

Problem 13–8. The force vector for the small tug is $3200(\cos\theta, \sin\theta)$, where θ is the angle to the left of the larger ships bow. The force vector for the large tug in this same coordinate scheme is $4000(\cos(-30^\circ), \sin(-30^\circ))$. The angle θ must be chosen so that the second component of the added forces is zero. Hence $3200\sin\theta = -4000\sin(-30^\circ)$, so $\theta = \arcsin(-4000\sin(-30^\circ)/3200)$, in radians.

Problem 13–9. The gravitational force of $(0,-250)$ is to be expressed in terms of the vectors $(\cos\theta, \sin\theta)$ along the incline and $(\cos(90+\theta), \sin(90+\theta))$ perpendicular to the incline. The magnitude of the force perpendicular to the incline is $|250\sin(90+\theta)|$. The value of θ which makes this equal to $250/6$ can be easily determined.

Problem 13–10. The true velocity of the plane is $250(\cos 40^\circ, \sin 40^\circ)+40(1,0)$. The ground speed of the plane is the magnitude of this vector, while the true course is the angle this vector makes with the vertical direction.

Problem 13–11. The slope m of the line should be chosen to minimize $(3-m)^2+(-4-(-3)m)^2+(2-2m)^2+(5-4m)^2$, which is the distance from the point $(3,-4,2,5)$ to the line with parametric equation $m(1,-3,2,4)$. Now proceed by translating the geometric picture into algebraic conditions.

Problem 13–13. How is R related to the function which rotates the point (x,y) in the plane through an angle of $\pi/6$ with the origin as a pivot?

Solutions to Exercises

Exercise 13–1. $\| v \|$ is the distance from the point v to the origin.

Exercise 13–2. $(3, -1, 8, 11)$ and $(3, 0, -9, 15)$.

Exercise 13–3. The line through the origin and the point $(2, 3)$.

Exercise 13–4. The vector is $(\cos\theta, \sin\theta)$.

Exercise 13–5. The parametric equation is $t(2, -1)$.

Exercise 13–6. Yes, since the two sets $\{s(4, -2) : s \in \mathbf{R}\}$ and $\{t(2, -1) : t \in \mathbf{R}\}$ are the same. This means that the parametric equation of a line is not unique.

Exercise 13–7. $(2, 5) + t(-1, -2)$, which is the same line as before.

Exercise 13–8. One form is $(1, 2, 3, 4) + t(1, 1, 1, 1)$. There are many others.

Exercise 13–9. $(1, 2, 3, 4) + t(1, 0, 0, -1)$.

Exercise 13–10. $\{(1, 2, 3) + t(3, 3, 3) + s(2, 0, -2) : t \in \mathbf{R} \text{ and } s \in \mathbf{R}\}$.

Exercise 13–11. By choosing $(4, 5, 6)$ as the base point, $(4, 5, 6)+t(-3, -3, -3)+s(-1, -3, -5)$, and by choosing $(3, 2, 1)$ as the base point, $(3, 2, 1) + t(-2, 0, 2) + s(1, 3, 5)$. Can you find two others?

Exercise 13–12. One choice is $P(x, y) = (4, 5, 6)+x(-3, -3, -3)+y(-1, -3, -5) = (4 - 3x - y, 5 - 3x - 3y, 6 - 3x - 5y)$. There are many others.

Exercise 13–13. $(1, 2)\bullet(3, 4) = 3 + 8 = 11$ and $(1, 2, 3, 4)\bullet(-1, 2, -2, 5) = -1 + 4 - 6 + 20 = 17$. Since $(1, 2, 3)\bullet(-3, 2, -1) = -3 + 4 - 3 = -2$, these two vectors are not perpendicular.

Exercise 13–14. Here $a = 11/10$ and $b = 1$, so the best fitting line is $y = (11/10)x + 1$.

Exercise 13–15. $-100 \sin 30^\circ(\cos 30^\circ, \sin 30^\circ)$. In which direction is this force pointing?

§14. Conclusion

Throughout these notes the principal objective has been to study the relationship that exists between certain specified variables. The final form of this relationship was usually expressed in terms of an equation relating the values of the variables. This equation was found in one or more of the following ways:

(1) from geometric reasoning applied to a picture of the relationship,

(2) from computing the same quantity in two different ways,

(3) from a verbal description of the relationship, or

(4) from an examination of experimental data.

Often the relationship between the variables took the form of a functional relationship. Because of this, some of the commonly used functions and their properties were studied.

Once the relationship between the variables was expressed algebraically, interesting properties of the relationship could be obtained by algebraic operations. These properties could then be interpreted in terms of the original variables.

This two step process of expressing relationships mathematically and interpreting the result of mathematical manipulations in the context of the original variables is the core of what makes mathematics useful.

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